

Different Differences

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From calculus we know that a derivative of a function $f(x)$ can be approximated using a difference quotient. There are different forms of the difference quotient, such as the forward difference (most common), backward difference and centered difference. I will introduce and discuss “Mickens differences,” which are decidedly different differences for approximating the derivatives in differential equations. Professor Ronald Mickens is an African-American Physics Professor at Clark Atlanta University who has written nearly 150 journal articles on this topic. These nonstandard finite differences can produce discrete solutions to a wide variety of differential equations with improved accuracy over standard numerical techniques. Applications drawn from first-semester Calculus to advanced computation fluid dynamics will be given.

Students are very welcome to attend. Knowledge of elementary derivatives/anti-derivatives and Taylor Approximations will be assumed.

OUTLINE

1. Approximating derivatives
2. Being discrete
3. A simple example (initial value problem)
4. A harder example ($m = 0$ boundary value problem)
5. Another example ($m > 0$ boundary value problem)
6. Ending with a (sonic) boom

What's the difference between a curve and a line? (This is not a "trick" question)

The name of the quantity which determines whether a graph will be a line or a curve is called the _____

We can find the slope of a line by computing $\frac{\Delta y}{\Delta x}$

We can find the slope of a curve by ... ?

The forward difference formula for $f'(x)$ is given by

$$f'(x) \approx \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

The backward difference formula is

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

The centered difference formula is

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

One way to show that these formulas “work” is to apply Taylor Expansions...

If a function $f(x)$ is infinitely differentiable at a point $x = a$ then the **Taylor Expansion** for the value $f(t)$ *about* the point $(a, f(a))$ is given by...

$$f(t) \approx f(a) + f'(a)(t-a) + f''(a)\frac{(t-a)^2}{2} + \dots$$

With a change of variables $t \rightarrow x+h$ and $a \rightarrow x$

$$f(x+h) \approx f(x) + f'(x)h + f''(x)\frac{h^2}{2} + \dots$$

$$f(x-h) \approx f(x) - f'(x)h + f''(x)\frac{h^2}{2} + \dots$$

Discrete Analogue

Split up an interval $a \leq x \leq b$ into N equal pieces, so

$$h = \frac{b - a}{N} \text{ and } x_k = a + kh \text{ for } k = 1, 2, \dots, N$$

Let

$$\begin{aligned} u_k &= f(x_k) \\ u_{k+1} &= f(x_{k+1}) = f(x_k + h) \\ u_{k-1} &= f(x_{k-1}) = f(x_k - h) \end{aligned}$$

Discrete Forward Difference

$$\frac{du}{dx} \approx \frac{u_{k+1} - u_k}{h}$$

Discrete Backward Difference

$$\frac{du}{dx} \approx \frac{u_k - u_{k-1}}{h}$$

Discrete Centered Difference

$$\frac{du}{dx} \approx \frac{u_{k+1} - u_{k-1}}{2h}$$

We can use these formulas to approximate derivatives in differential equations to produce difference equations

Exponential Example

Consider the initial value problem (IVP)

$$\frac{dy}{dx} = y, \quad y(0) = 1$$

We know the exact solution is $y(x) =$

The discrete version of the exact solution is $y_k =$

We can solve the IVP by **discretizing** the initial value problem.

Using a standard finite-difference scheme the discrete form of the IVP becomes

$$\frac{y_{k+1} - y_k}{h} = y_k, \quad \text{for } k = 1, 2, \dots, N \text{ and } y_0 = 1$$

which when rearranged or solved becomes

$$y_{k+1} - y_k = y_k h \Rightarrow y_{k+1} = y_k + h y_k = y_k(1 + h)$$

Applying the initial condition at $k=0$

$$y_1 = y_0(1 + h) = 1 + h$$

when $k = 1$

$$y_2 = y_1(1 + h) = (1 + h)(1 + h) = (1 + h)^2$$

when $k = 2$

$$y_3 = y_2(1 + h) = (1 + h)(1 + h)^2 = (1 + h)^3$$

Therefore

$$y_k = (1 + h)^k, \quad k = 0, 1, 2, \dots, N$$

Numerical Error

How accurate was the solution generated by the standard finite difference scheme?

The exact solution to the differential equation (ODE) $y' = y, y(0) = 1$ is

$$y(x) = e^x$$

which has a discrete analogue given by

$$y_k = e^{kh}$$

The solution to the related difference equation (OΔE) was

$$y_k = (1 + h)^k, k = 0, 1, 2, \dots, N$$

The error ϵ_k at any point $x_k = kh$ is given by

$$\epsilon_k = y(x_k) - y_k = e^{hk} - (1 + h)^k$$

At $k = 0$ there is no error:

$$\epsilon_0 = e^0 - (1 + h)^0 = 1 - 1 = 0$$

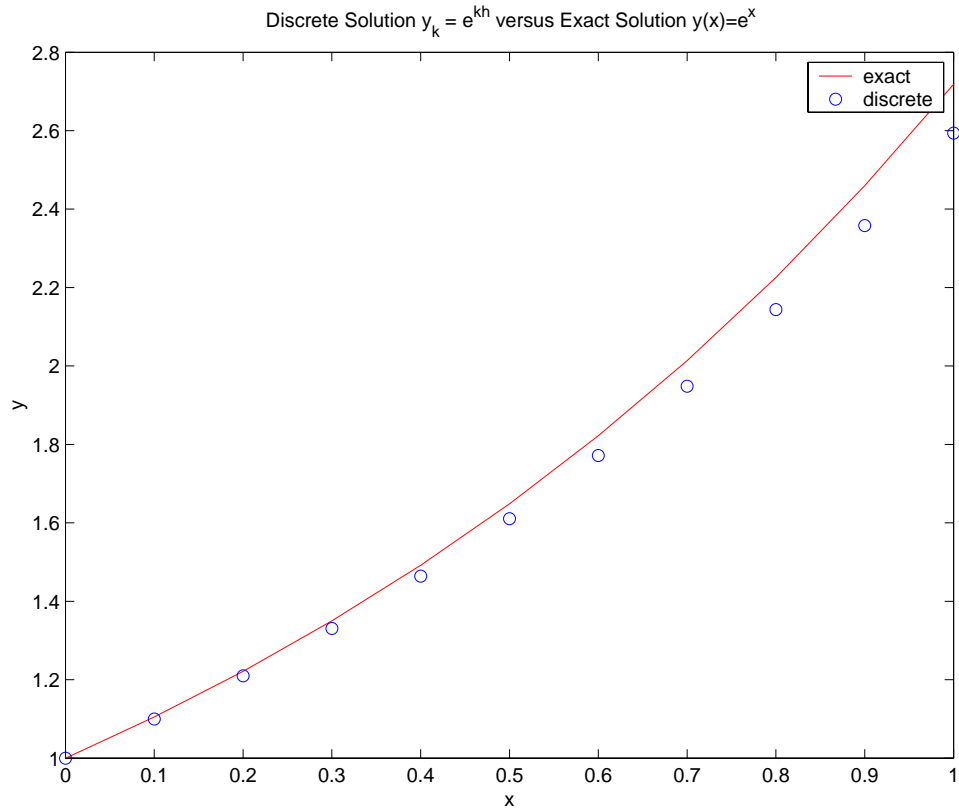
At $k = 1$

$$\begin{aligned}\epsilon_1 &= e^h - (1 + h)^1 = \left(1 + h + \frac{h^2}{2} + \dots\right) - (1 + h) \\ &= \frac{h^2}{2} + \dots\end{aligned}$$

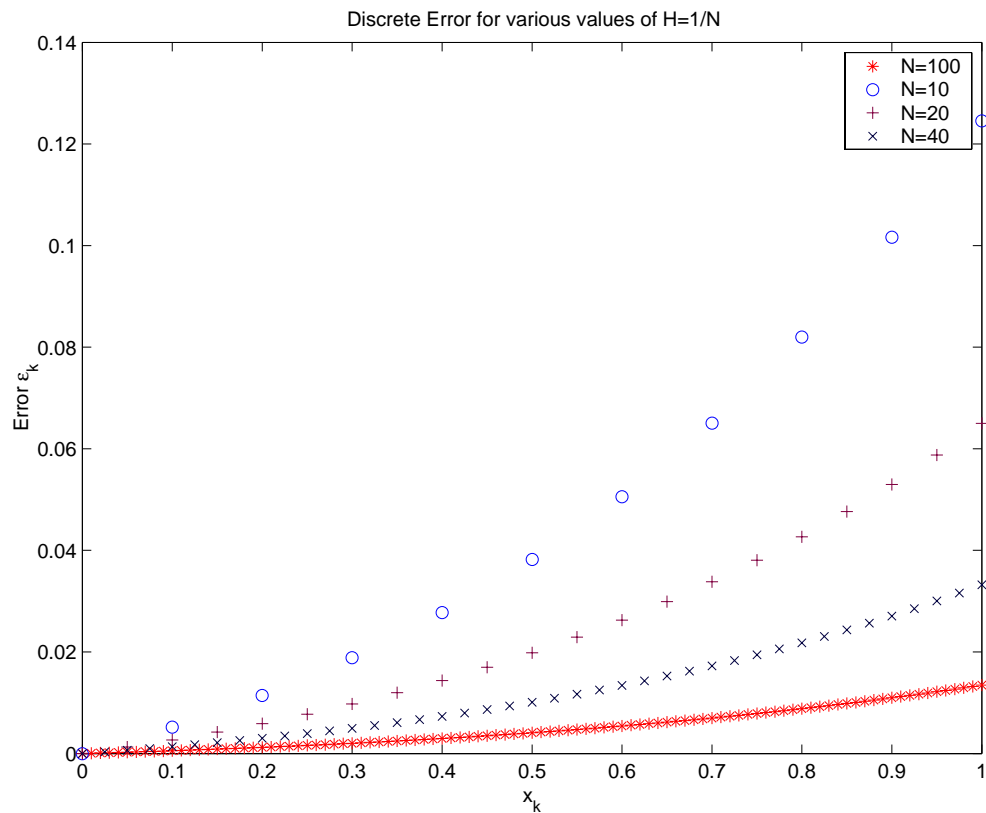
At $k = 2$

$$\begin{aligned}\epsilon_2 &= e^{2h} - (1 + h)^2 = \left(1 + 2h + \frac{(2h)^2}{2} + \dots\right) - (1 + h)^2 \\ &= (1 + 2h + 2h^2 + \dots) - (1 + 2h + h^2) \\ &= h^2 + \dots\end{aligned}$$

Numerical Results for $N = 10$



Discrete Error as N Increases (h decreases)



Different Differences

Professor Ronald Mickens of Clark Atlanta University has suggested a different way to approximate the derivative

$$f'(x) \approx \frac{f(x + \phi_1(h)) - f(x)}{\phi_2(h)}, \quad \text{where } \phi_n = h + \dots$$

Note that as $h \rightarrow 0$ the above difference quotient yields $f'(x)$ exactly as the standard formulae do.

The discrete analogue of Mickens' suggestion is

$$\frac{dy}{dx} \approx \frac{y_{k+1} - \psi y_k}{\phi(h)}, \quad \text{where } \psi = 1 + \dots \text{ and } \phi(h) = h + \dots$$

The beauty of this idea is that it gives us more flexibility to tailor our approximation technique to the particular differential equation we're trying to discretize.

Most often $\psi = 1$ and we need to choose a **denominator function** $\phi(h)$

$$\phi(h) = \begin{cases} h, \\ \sin(h), \\ e^h - 1, \\ 1 - e^{-h}, \\ h \\ \frac{1 - h}{1 - e^{-\lambda h}}, \\ \frac{\lambda}{\lambda} \\ \vdots \end{cases}$$

Application of a Mickens Difference

Suppose we reconsider the ODE $\frac{dy}{dx} = y$, $y(0) = 1$. How do we make our choice of denominator function? There are no firm rules which direct you in every case. In this simple example we know the exact solution looks exponential so we should try a choice with this functional behavior

$$\phi(h) = e^h - 1$$

Our related difference equation (O Δ E) would become

$$\frac{y_{k+1} - y_k}{e^h - 1} = y_k, \quad y_0 = 1$$

which can be rearranged to be

$$y_{k+1} = y_k + \phi(h)y_k = y_k + y_k(e^h - 1) \Rightarrow y_{k+1} = e^h y_k$$

Applying the initial condition at $k = 0$

$$y_1 = y_0 e^h$$

When $k = 1$,

$$y_2 = y_1 e^h = e^h e^h = e^{2h}$$

When $k = 2$,

$$y_3 = y_2 e^h = e^h e^{2h} = e^{3h}$$

Therefore,

$$y_k = e^{kh}, \quad k = 0, 1, 2, \dots, N$$

is the discrete version of the solution to the ODE produced using the Mickens finite difference scheme.

Error due to the Mickens Difference Method

Recall that the error ϵ_k at any point $x_k = kh$ is given by

$$\epsilon_k = y(x_k) - y_k$$

The exact solution to the ODE is $y(x) = e^x$

The solution to the difference equation generated by using a standard finite-difference discretization of the ODE was $y_k = (1 + h)^k$

The solution to the difference equation generated by using the nonstandard discretization of the ODE is $y_k = e^{kh}$.

Thus the numerical error of the Mickens scheme is given by

$$\epsilon_k = e^{x_k} - e^{kh} = e^{kh} - e^{kh} = 0$$

In other words, by making a good choice of denominator function one can produce a difference equation which represent an **exact** discrete version of the solution of the differential equation.

We have been able to “approximate” the differential equation exactly!

Was this a fluke? No!

Applications of Mickens Differences

Consider the following boundary value problem in cylindrical coordinates for the function $u(r)$

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) - m^2 u &= 0, & m \text{ constant} \\ r \frac{du}{dr} \Big|_{r=0} &= S, \\ u(1) &= G. \end{aligned}$$

When $m = 0$ the ODE becomes

$$\frac{d}{dr} \left(r \frac{du}{dr} \right) = 0$$

with the conditions

$$u(r) = G \text{ at } r = 1 \text{ and } r \frac{du}{dr} = S \text{ at } r = 0$$

Recall, if $\frac{d}{dr} (\quad) = 0 \leftrightarrow (\quad) = \text{constant}$

Therefore $r \frac{du}{dr} =$

The exact solution $u(r)$ of the boundary value problem is

$$u(r) = S \log(r) + G$$

Check:

Applying Standard Finite Differences

We can write our boundary value problem from before as the initial value problem that we actually solved and then use our discretization technique....

$$r \frac{du}{dr} = S, \quad u(1) = G$$

First we split up the interval $0 < r_0 \leq r \leq 1$ into N pieces, so $r_k = r_0 + kh$, $k = 0, 1, 2, \dots, N$ and $h = \frac{1 - r_0}{N}$

The discrete version of the ODE using standard differences will be

$$r_k \frac{u_{k+1} - u_k}{h} = S, \quad u_N = G$$

which can be rearranged to produce

$$u_k = u_{k+1} - \frac{Sh}{r_k}, \quad k = 0, 1, 2, \dots, N - 1$$

We can find every value of u_k on the grid by starting at $k = N$ since $u_N = G$.

Then u_{N-1} can be computed in terms of u_N , and u_{N-2} can be computed in terms of u_{N-1} and so on.

This process is called a **marching scheme**.

Applying Nonstandard Finite Differences

Consider again the ODE

$$r \frac{du}{dr} = S, \quad u(1) = G$$

“Buckmire’s Method”

By manipulating the differential equation and approximating the derivatives

$$r \frac{du}{dr} = \frac{du}{\frac{dr}{r}} = \frac{du}{d(\log(r))} \approx \frac{\Delta u}{\Delta(\log(r))}$$

Δu is defined as $u_{k+1} - u_k$ and

$\Delta \log(r)$ is $\log(r_{k+1}) - \log(r_k)$.

The discrete version of the ODE using Mickens differences will be

$$\frac{u_{k+1} - u_k}{\log(r_{k+1}) - \log(r_k)} = S \text{ for } k = 0, 1, \dots, N-1, \text{ with } u_N = G$$

which can be rearranged to be

$$\begin{aligned} u_k &= u_{k+1} - S[\log(r_{k+1}) - \log(r_k)] \\ &= u_{k+1} - S \log(r_{k+1}/r_k) \end{aligned}$$

This is also a marching scheme for determining all values of u_k from $k = N - 1, N - 2, \dots, 1, 0$ with $u_N = G$

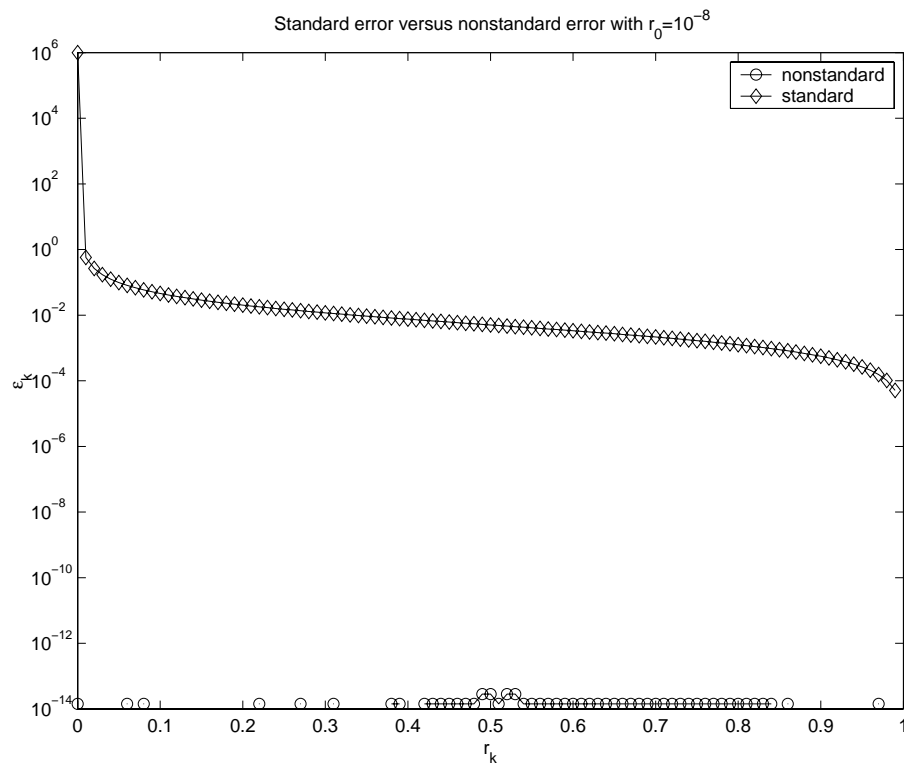
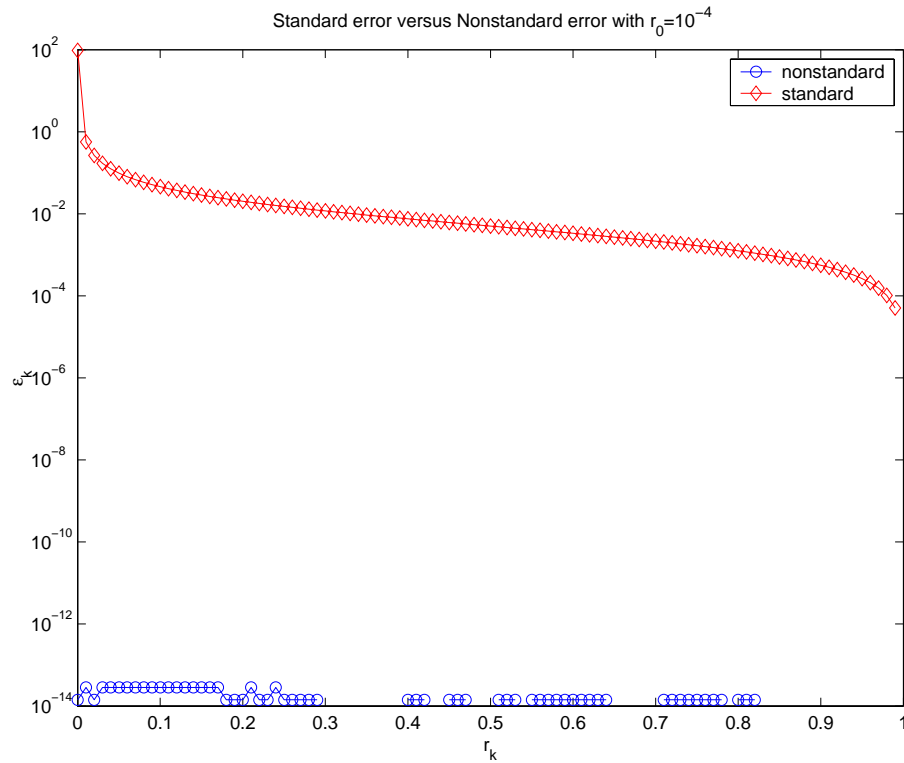
How do the two competing numerical methods compare?

Numerical Results for $m = 0$ case

Let $S = G = 1$ and choose $r_0 = 10^{-4}$ and $N = 100$.

Then $h = \frac{1-10^{-4}}{100}$

We know the exact solution will be $u(r) = S \log r + G$



The $m > 0$ problem

Recall the boundary value problem is

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) - m^2 u &= 0, & m \text{ constant} \\ r \frac{du}{dr} \Big|_{r=0} &= S, \\ u(1) &= G. \end{aligned}$$

We can simplify the derivative terms to obtain

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - m^2 u = 0$$

which becomes

$$r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - m^2 r^2 u = 0$$

If we let $z = mr$ then this equation can be transformed into

$$z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} - z^2 u = 0$$

This is known as the modified Bessel's Equation of zeroth order.

It's such a well-known equation that its solutions $u(z)$ are functions called the **modified Bessel's functions of the first and second kind** $K_0(z)$ and $I_0(z)$

There are numerous functions whose name we know who are really just solutions of a differential equation. For example, the equation

$$\frac{d^2u}{dz^2} + u = 0$$

has two famous solutions: _____ and _____

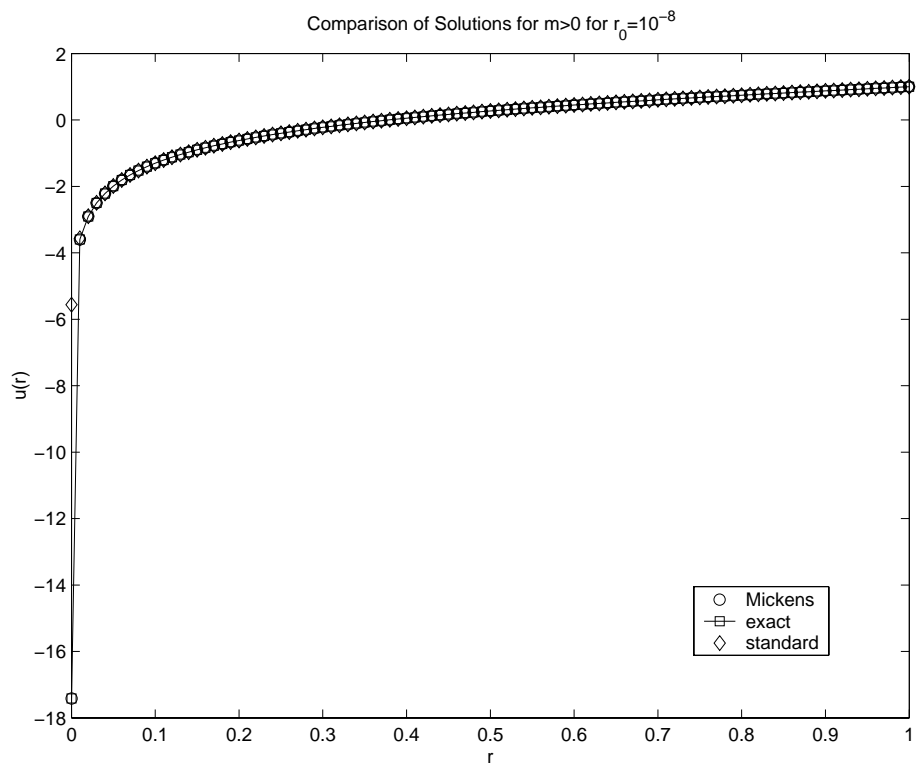
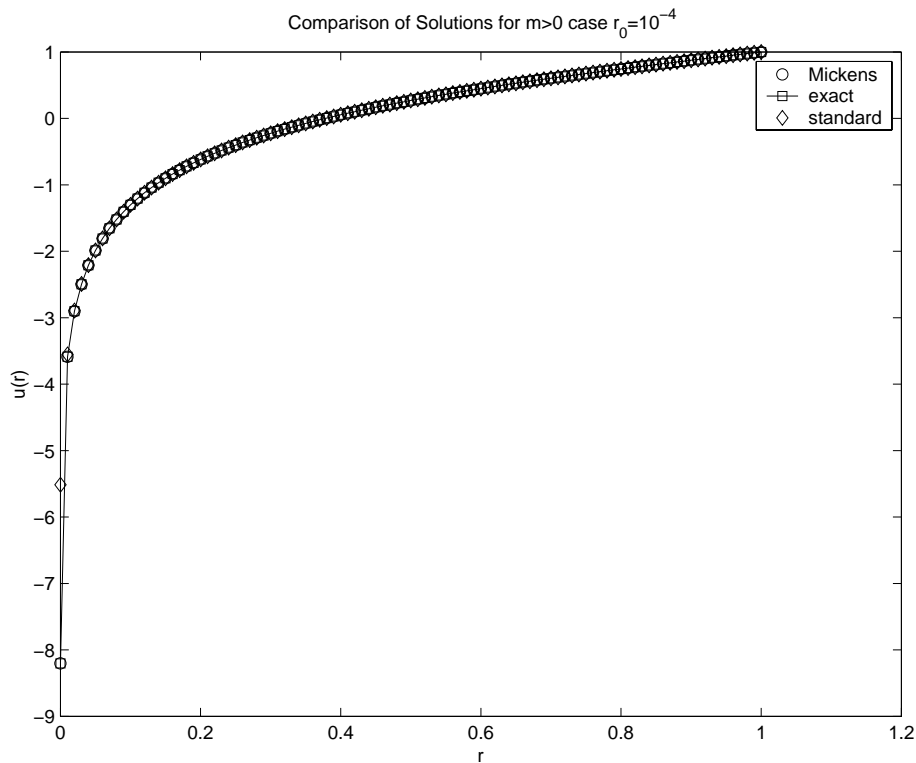
When $m > 0$ the exact solution to our boundary value problem can be written in terms of $I_0(mr)$ and $K_0(mr)$

$$u(r) = -SK_0(rm) + (G + SK_0(m))\frac{I_0(rm)}{I_0(m)}$$

Since we have an exact solution we can compare it to numerical results generated from using standard finite difference approximations to the modified Bessel's equation versus using a nonstandard (Buckmire) finite difference approximation

Comparing Results for $m > 0$

Let $S = G = 1$ and choose $N = 100$ and $r_0 = 10^{-4}$ or 10^{-8} . Then $h = \frac{1-r_0}{100}$



The Problem We Really Want To Solve (Theoretical Aerodynamics / Computational Fluid Dynamics)

The Kármán-Guderley equation

$$(K - (\gamma + 1)\phi_x)\phi_{xx} + \phi_{\tilde{r}\tilde{r}} + \frac{1}{\tilde{r}}\phi_{\tilde{r}} = 0. \quad (1)$$

Inner boundary condition

$$\begin{aligned} \phi(x, \tilde{r}) &\rightarrow S(x) \log \tilde{r} + G(x), & \text{as } \tilde{r} \rightarrow 0, \quad |x| \leq 1 \\ \phi(x, \tilde{r}) &\text{ bounded,} & \text{for } \tilde{r} = 0, \quad |x| > 1. \end{aligned} \quad (2)$$

Outer boundary condition

$$\phi(x, \tilde{r}) \rightarrow \frac{\mathcal{D}}{4\pi} \frac{x}{(x^2 + K\tilde{r}^2)^{3/2}}, \quad \text{as } (x^2 + \tilde{r}^2)^{1/2} \rightarrow \infty. \quad (3)$$

Future Work: Problems We Haven't Solved (Yet!)

The Bratu Problem

$$\Delta u + \lambda e^u = 0, \quad u = 0 \text{ on } \partial U$$

In one-dimension the problem becomes simpler

$$\frac{d^2}{dx^2}u(x) + \lambda e^{u(x)} = 0, \quad u(0) = u(1) = 0$$

and there is an exact solution

$$u(x) = -2 \ln \left(\frac{\cosh[(x - \frac{1}{2})\frac{\theta}{2}]}{\cosh(\frac{\theta}{4})} \right)$$

which satisfies the boundary conditions

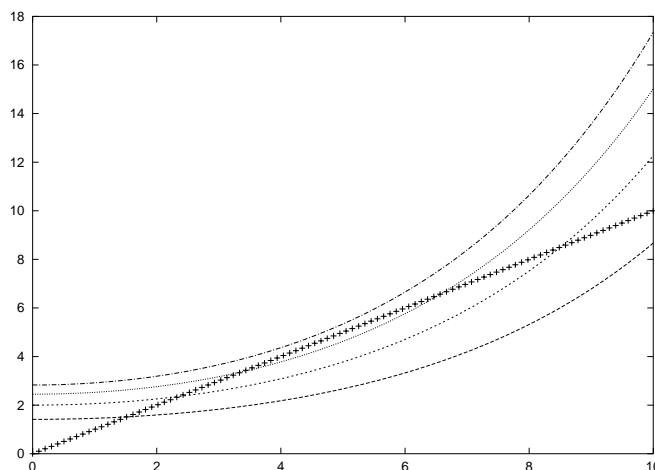
$$u(0) =$$

and

$$u(1) =$$

and satisfies the differential equation if

$$\theta = \sqrt{2\lambda} \cosh \left(\frac{\theta}{4} \right)$$



Bratu-Gel'fand Problem

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) + \lambda e^{u(r)} = 0, \quad u(1) = 0 \text{ and } u(0) < \infty$$

Joint Work with Mickens

$$\frac{\partial T}{\partial t} = \frac{1}{10} \frac{\partial^2(T^{5/2})}{\partial r^2} + \frac{1}{10r} \frac{\partial^2(T^{5/2})}{\partial r^2} + (1 - T^2)(cT^2 - T^{1/2})$$

$$T(1, t) = 0$$

$$T(r, 0) = A\left(r + \frac{1}{8}\right)(r - 1)^2$$