Introduction to Fractional Differentiation

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Outline

History of Fractional Differentiation

We start our problem with the linear function $f(x) = x$ and ask the question what is the nth derivative of the function.

$$\frac{D^n x}{Dx^n}$$

"An apparent paradox, from which one day useful consequences will be drawn." -Gottfried Wilhelm Leibniz 1695

Nth order Derivative

Deriving the nth order derivative of $f(x) = (x-a)^k$.

$$f'(x) = k(x-a)^{k-1}$$

$$f''(x) = k(k-1)(x-a)^{k-2}$$

Nth Derivative of the function $f(x) = (x-a)^k$

$$f^{(n)}(x) = (x-a)^{k-n}\frac{k!}{(k-n)!} \tag{1}$$

Gamma Function

The Gamma Function is an extension of the factorial function, defined by this integral.

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$$

Important results

$\Gamma(n) = (n-1)!$ When n $\in \mathbb{Z}$ $\quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\quad \Gamma(\frac{1}{2} + n) = \frac{2n!}{4^n n!}\sqrt{\pi}$

Nth Derivative Using Gamma Functions

$$f^{(n)}(x) = (x-a)^{k-n} \frac{\Gamma(k+1)}{\Gamma(k-n+1)}$$

Beta Function and the Beta-Gamma Relation

β Function

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

In order to show the Beta-Gamma Relation, I will introduce 2 Gamma functions multiplied together.

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u}u^{x-1}du \int_0^\infty e^{-v}v^{y-1}dv$$

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v}u^{x-1}v^{y-1}dudv$$

Changing Variables and the Jacobian

In order to change variables in any scenario, the Jacobian must be introduced. The Jacobian is the determinant consisting of the partial derivatives of the new functions. If the Jacobian is non-zero, the change of variables can be executed. Take for example. $u = u(\xi, \eta)$, $x = (\xi, \eta)$, and $y = (\xi, \eta)$

$$|J(z, t)| = \left\{ \begin{array}{cc} \xi_x & \xi_y \\ \eta_x & \eta_y \end{array} \right\} = (\xi_x \eta_y) - (\eta_x \xi_y) \neq 0$$

The Beta-Gamma Relation

Changing variables u=f(z,t)=zt, v=g(z,t)=z(1-t)

$$\Gamma(x)\Gamma(y) = \int_{z=0}^{\infty}\int_{t=0}^{1} e^{-z}(zt)^{x-1}(z(1-t))^{y-1}|J(z,t)|dtdz$$

$$\Gamma(x)\Gamma(y) = \int_{z=0}^{\infty}\int_{t=0}^{1} e^{-z}(zt)^{x-1}(z(1-t))^{y-1}zdtdz$$

THe Beta Gamma Relation

$$\Gamma(x)\Gamma(y) = \int_{z=0}^{\infty} e^{-z} z^{x+y-1} dz \int_{t=0}^{1} t^{x-1}(1-t)^{y-1} dt$$

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)\beta(x,y)$$

Finally we arive at our desired relation.

The β,Γ Relation

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2}$$

The Fractional Derivative

There are many definitions of the fractional derivative, and the one I will be discussing is the Grünnwald-Letnikov definition. The operator acts as an integral operator. It is evaluated on an interval of points ranging from a to t, making fractional differentiation a global operator on the function. This was the original Paradox that Leibniz mentioned.

Grünnwald-Letnikov Fractional Derivative

$$ {}_aD_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}f(\tau)d\tau \tag{3}$$

General Form of a Fractional Derivative

Using the definition of fractional derivatives for the function $f(\tau) = (\tau - a)^r$, we get.

$$_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} (\tau - a)^r d\tau$$

Further manipulating this into a general form,

$$\text{Let } \tfrac{\tau - a}{t - a} = \theta$$

$$t - \tau = (t-a) - (\tau - a) = (t-a)(1-\theta)$$

General Form of a Fractional Derivative

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} (t-a)^{p+r} \int_0^1 (1-\theta)^{p-1} \theta^r \, d\theta $$

Now we have a β function inside the integral.

$$ \int_0^1 (1-\theta)^{p-1} \theta^r \, d\theta = \int_0^1 \beta(p, r+1) = \frac{\Gamma(p)\Gamma(r+1)}{\Gamma(p+r+1)} $$

$$ {}_aD_t^{-p} f(t) = \frac{\Gamma(p)\Gamma(r+1)}{\Gamma(p+r+1)} \frac{1}{\Gamma(p)} (t-a)^{p+r} $$

The $-p^{th}$ order fractional derivative of f(t)=(t-a)^r

$$ (t-a)^{p+r} \frac{\Gamma(r+1)}{\Gamma(p+r+1)} \tag{4} $$

Comparing the 2 Derivatives

Integer Derivative

$$f^{(p)}(t) = (t-a)^{p-n}\frac{\Gamma(p+1)}{\Gamma(p-n+1)}$$

Fractional Derivative

$${}_aD_t^{-p}f(t) = (t-a)^{p+r}\frac{\Gamma(r+1)}{\Gamma(p+r+1)}$$

Numerical Example

Take for example, the half-derivative of the function $f(x) = x^2$.

$$f^{(\frac{1}{2})}(x^2) = (x-0)^{2+\frac{1}{2}} \frac{\Gamma(2+1)}{\Gamma(2+\frac{1}{2}+1)}$$

$$f^{(\frac{1}{2})}(x^2) = x^{\frac{5}{2}} \frac{\Gamma(3)}{\Gamma(\frac{7}{2})}$$

Numerical Example

$$\Gamma(n) = (n-1)! \quad \text{When n} \in \mathbb{Z} \qquad \Gamma(\tfrac{1}{2}+n) = \tfrac{2n!}{4^n n!}\sqrt{\pi}$$

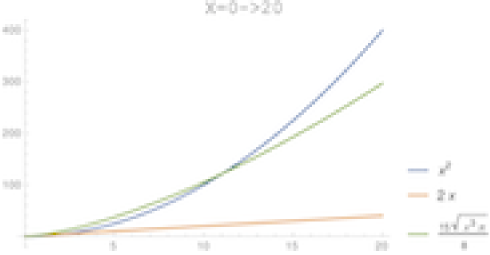
Using the Γ function results discussed earlier, the function can be written in a simple factorial form, which is easy to be evaluated.

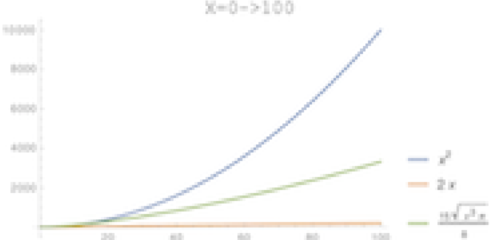
$$f^{(\frac{1}{2})}(x^2) = x^{\frac{5}{2}} \frac{\Gamma(3)}{\Gamma(\frac{7}{2})} = x^{\frac{3}{2}} \frac{6!}{4^3 3!}\sqrt{\pi}$$

The Half-Derivative of x^2

$$f^{(\frac{1}{2})}(x^2) = \frac{15}{8}\sqrt{x^3 \pi}$$

Numerical Example





Other Examples: Constant

Now say that $f(t) = c$, where c is an arbitrary constant. One would think, that like an integer order derivative, that the result would be zero for a fractional derivative, but this is not the case.

$$_aD_t^{-p}c = \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} c d\tau$$

$$_aD_t^{-p}c = \frac{c}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} d\tau$$

Performing a u-sub.

$$u = t - \tau$$

$$-du = d\tau$$

$$_aD_t^{-p}c = \frac{c}{\Gamma(p)}\int_a^t -u^{p-1} d\tau$$

Other Examples: Constant

$$ {}_aD_t^{-p}c = \frac{c}{\Gamma(p)} \frac{-(t-\tau)^p}{p} \Big|_{\tau=t}^{\tau=a} $$

Which leaves us with,

$$ {}_aD_t^{-p}c = \frac{c}{\Gamma(p)} \frac{(t-a)^p}{p} $$

So the fractional derivative of a constant is another constant, depending on the points evaluated and the order of the fractional derivative.

Other Examples: Trig Functions

To break down the general nth derivatives of trig functions, we start with the exponential.

$$\frac{d^v}{dx^v} e^{ax} = a^v e^{ax}$$

Now if we discuss trig functions, we know we can express trig functions in terms of exponentials from Euler's formula.

$$e^{ix} = \cos x + i \sin x$$

Other Examples: Trig Functions

So take for example the v^{th} derivative of $\cos x$.

$$\frac{d^v}{dx^v}\cos x = \frac{d^v}{dx^v}\frac{e^{ix} - e^{-ix}}{2} = \frac{i^v e^{ix} + (-i)^v e^{-ix}}{2}$$

Using Euler's formula, let's solve for i.

$$x = \frac{\pi}{2}$$

$$e^{ix} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$$

$$(e^{i\frac{\pi}{2}})^n = (i)^n$$

$$(\pm i)^n = (\pm e^{i}\frac{\pi}{2})^n$$

$$\frac{d^v}{dx^v}\cos x = \cos(x + v\frac{\pi}{2})$$

Properties

Linearity of a derivative is expressed below.

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

This property also applies to fractional derivatives. **Compositions** of derivatives would be taking the derivative of a derivative. Here is the compositon of an integer derivative.

$$\frac{d^n y}{dx^n}(\frac{d^m y}{dx^m}) = \frac{d^{m+n} x}{dy^{m+n}}$$

Below is a proof that will show the same result for a fractional derivative or,

$${}_aD_t^{-q}({}_aD_t^{-p}f(t)) =_a D_t^{-q-p}f(t)$$

$${}_aD_t^{-q}({}_D{}_t^{-p}f(t)) = \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}({}_aD_t^{-p}f(\tau))d\tau$$

Composition of Fractional Differentiation

$$ {}_aD_t^{-q}({}_aD_t^{-p}f(t)) = \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}\left[\frac{1}{\Gamma(p)}\int_a^\tau (\tau-\xi)^{p-1}f(\xi)d\xi\right]d\tau $$

$$ {}_aD_t^{-q}({}_aD_t^{-p}f(t)) = \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t \left[\int_\xi^t (t-\tau)^{q-1}(\tau-\xi)^{p-1}d\tau\right]f(\xi)d\xi $$

We now perform a U-Substitution for the integrals.

$$ \tau-\xi = \theta(t-\xi) $$

$$ t-\tau = t-\xi-(\tau-\xi) $$

$$ t-\tau = (t-\xi)(1-\theta) $$

$$ \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t (t-\xi)^{p+q-1}f(\xi)\int_0^1 (1-\theta)^{q-1}(\tau-\xi)^{p-1}\theta^{p-1}d\theta d\xi $$

Composition of Fractional Differentiation

We have a β function in the second integral, so this integral now becomes,

$$ {}_aD_t^{-q}({}_aD_t^{-p}f(t)) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t (t-\xi)^{p+q-1} f(\xi)\beta(p,q) $$

We know that a β function can be written in terms of Γ functions.

$$ {}_aD_t^{-q}({}_aD_t^{-p}f(t)) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t (t-\xi)^{p+q-1} f(\xi)\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} $$

Finally we arrive at,

$$ {}_aD_t^{-q}({}_aD_t^{-p}f(t)) = \frac{1}{\Gamma(p+q)} \int_a^t (t-\xi)^{p+q-1} f(\xi) = {}_a D_t^{-q-p} f(t) $$