Introduction to Singular Perturbation Theory

Erika May

Department of Mathematics
Occidental College

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Outline

Introduction

- Real world problems contain parameters that mimic real situations that change the nature of the problem
- Perturbation Theory
  - **Regular perturbation** happens when the problem where the parameter ε is small but nonzero is qualitatively the same as the problem where ε is zero
  - **Singular perturbation** happens when the problem where ε is small but nonzero is qualitatively different than the problem where ε is zero

    $\Rightarrow$ Bifurcation
  - Approximate using power series expansion in ε

Perturbation Theory

- Regular perturbation example:

$$x^2 - x + \varepsilon = 0$$

Exact solution:

$$x = \frac{1 \pm \sqrt{1 - 4\varepsilon}}{2}$$

Let $\varepsilon = 0$:

$$x^2 - x = 0 \Rightarrow x = 0, 1$$

- Singular perturbation example:

$$\varepsilon x^2 + 2x + 1 = 0$$

Exact solution:

$$x = \frac{-2 \pm \sqrt{4 - 4\varepsilon}}{2\varepsilon}$$

Let $\varepsilon = 0$:

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

Asymptotic Expansion

- Straightforward asymptotic expansion:
  (i) Assume solutions of given function can be asymptotically expanded in ε using power series:

$$y = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots + y_n(x)\varepsilon^n + \mathcal{O}(\varepsilon^{n+1})$$

  (ii) Substitute expansion into original function
  (iii) Isolate zeroth order terms and solve

Singular Perturbation Theory

- Boundary layer problems
  - Interval of rapid change
  - Straightforward expansion using does not satisfy all boundary conditions
- Method of matched asymptotic expansions
  - Construct separate asymptotic expansions for inside and outside of boundary layer and create composite approximation

Example

Motivating example: **boundary value problem** of second-order, linear, constant coefficient ODE

$$\varepsilon y'' + 2y' + y = 0, \ x \in (0, 1)$$

$$y(0) = 0, \ y(1) = 1$$

$\Rightarrow$ This is a singular perturbation problem

Example: Exact Solution

$$\varepsilon y'' + 2y' + y = 0$$
$$y(0) = 0, \; y(1) = 1$$

Characteristic polynomial:

$$\varepsilon s^2 + 2s + 1 = 0$$

$$s_1 = \frac{-1 + \sqrt{1-\varepsilon}}{\varepsilon}, \; s_2 = \frac{-1 - \sqrt{1-\varepsilon}}{\varepsilon}$$

Thus, the general solution will be:

$$y(x, \varepsilon) = c_1 e^{s_1 x} + c_2 e^{s_2 x}$$

where c_1, c_2 are arbitrary constants. Imposing the boundary conditions at $x = 0$ and $x = 1$, the solution is:

$$y(x, \varepsilon) = \frac{e^{s_1 x} - e^{s_2 x}}{e^{s_1} - e^{s_2}}$$

Example: Boundary Layer

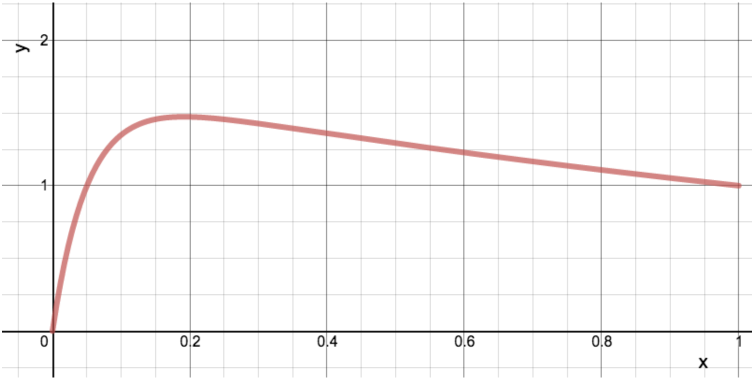

Figure 1: Exact solution with $\varepsilon = 0.1$

Example: Outer Expansion

$$\varepsilon y'' + 2y' + y = 0$$

$$y(0) = 0, \; y(1) = 1$$

Outer region varies slowly (unperturbed), so proceed with straightforward expansion:

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \mathcal{O}(\varepsilon^2)$$

$$\Downarrow$$

$$\varepsilon(y_0'' + \varepsilon y_1'' + \dots) + 2(y_0' + \varepsilon y_1' + \dots) + (y_0 + \varepsilon y_1 + \dots) = 0$$

Since boundary layer is at $x = 0$ and we're evaluating the outer region, impose boundary condition $y(1) = 1$ on expansion:

$$2y_0' + y_0 = 0$$

$$y_0(1) = 1$$

Example: Outer Expansion

Solve linear first-order ODE:

$$y_0(x) = ce^{sx}$$

$$2s + 1 = 0$$

$$y_0(x) = e^{\frac{1}{2}(1-x)}$$

Denote outer expansion as y_{outer}:

$$y_{outer} = e^{\frac{1}{2}(1-x)}$$

Example: Inner Expansion

$$\varepsilon y'' + 2y' + y = 0$$
$$y(0) = 0, \; y(1) = 1$$

To construct an inner expansion, rescale the narrow boundary layer using a stretching variable:

$$X = \frac{x}{\delta(\varepsilon)}$$

Seek inner solution:

$$Y(X, \varepsilon) = y(x, \varepsilon)$$

Chain rule gives us:

$$y' = \frac{dy}{dx} = \frac{dY}{dx} = \frac{dY}{dX}\frac{dX}{dx} = \frac{1}{\delta}\frac{dY}{dX} = \frac{1}{\delta}Y' \Rightarrow y'' = \frac{1}{\delta^2}Y''$$

Our original differential equation becomes:

$$\frac{\varepsilon}{\delta^2}Y'' + \frac{2}{\delta}Y' + Y = 0$$

Example: Inner Expansion

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{2}{\delta} Y' + Y = 0$$

After rescaling, we must determine correct two-term dominant balancing of terms. We have three coefficients:

$$\frac{\varepsilon}{\delta^2}, \qquad \frac{2}{\delta}, \qquad 1$$

Two options:

(a) $\frac{\varepsilon}{\delta^2}$ and 1 are of the same magnitude and dominant over $\frac{2}{\delta}$

(b) $\frac{\varepsilon}{\delta^2}$ and $\frac{2}{\delta}$ are of the same magnitude and dominant over 1

Example: Inner Expansion

$$\frac{\varepsilon}{\delta^2} \qquad \frac{2}{\delta} \qquad 1$$

Outcomes:

(a) $\frac{\varepsilon}{\delta^2} \sim 1$ implies $\delta(\varepsilon) = \mathcal{O}(\sqrt{\varepsilon})$:

$$1 \qquad \frac{2}{\sqrt{\varepsilon}} \qquad 1$$

Since ε is small, so no dominant balance

(b) $\frac{\varepsilon}{\delta^2} \sim \frac{2}{\delta}$ implies $\delta(\varepsilon) = \mathcal{O}(\varepsilon)$:

$$\frac{1}{\varepsilon} \qquad \frac{2}{\varepsilon} \qquad 1$$

Example: Inner Expansion

From dominant balance, we can let:

$$\delta(\varepsilon) = \varepsilon, \text{ and so } X = \frac{x}{\varepsilon}$$

New scaled differential equation:

$$\frac{\varepsilon}{\varepsilon^2} Y'' + \frac{2}{\varepsilon} Y' + Y = 0$$

$$\Downarrow$$

$$Y'' + 2Y' + \varepsilon Y = 0$$

$$Y(0) = 0$$

Example: Inner Expansion

Construct expansion:

$$Y(X,\varepsilon) = Y_0(X) + \varepsilon Y_1(X) + \mathcal{O}(\varepsilon^2)$$

$$\Downarrow$$

$$(Y_0'' + \varepsilon Y_1'' + \dots) + 2(Y_0' + \varepsilon Y_1' + \dots) + \varepsilon(Y_0 + \varepsilon Y_1 + \dots) = 0$$

Impose boundary condition at $X = 0$:

$$Y_0'' + 2Y_0' = 0$$

$$Y_0(0) = 0$$

Example: Inner Expansion

Solve second order ODE:

$$Y_0(X) = c_1 e^{s_1 X} + c_2 e^{s_2 X}$$

$$s^2 + 2s = 0$$

$$Y_0(X) = c(1 - e^{-2X})$$

Denote inner expansion as Y_{inner}:

$$Y_{inner} = c(1 - e^{-2X})$$

Example: Matching

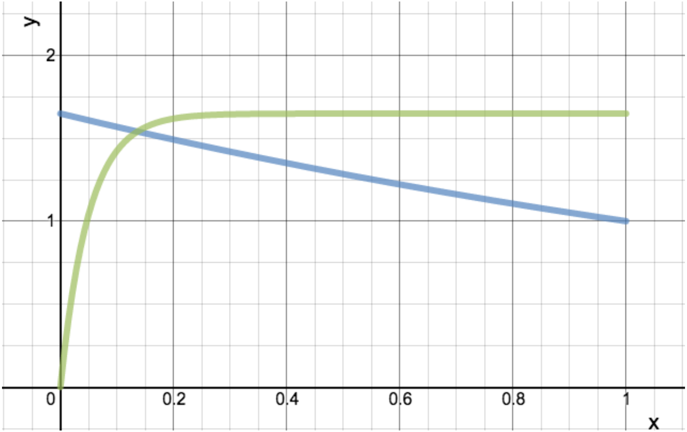

Figure 2: y_{outer} (blue) and Y_{inner} (green) at $\varepsilon = 0.1$

Example: Matching

Determine unknown constant c by "matching" inner and outer solution, given by the matching condition:

$$\lim_{X\to\infty} Y_{inner}(X) = \lim_{x\to 0^+} y_{outer}(x)$$

$$\lim_{X\to\infty} c(1 - e^{-2X}) = \lim_{x\to 0^+} e^{\frac{1}{2}(1-x)}$$

Which implies:

$$c = e^{1/2} = y_{overlap}$$

Final inner expansion, with $y(x,\varepsilon) = Y(X,\varepsilon)$ and $X = \dfrac{x}{\varepsilon}$:

$$y_{inner} = e^{1/2}(1 - e^{-2x/\varepsilon})$$

Example: Composite

Our composite approximation follows:

$$y_{composite} = y_{inner} + y_{outer} - y_{overlap}$$

Matching condition showed us $y_{overlap} = e^{1/2}$, so:

$$y_{composite} = [e^{1/2}(1 - e^{-2x/\varepsilon})] + [e^{1/2(1-x)}] - e^{1/2}$$

$$\Downarrow$$

$$y_{composite} = e^{\frac{1-x}{2}} - e^{\frac{\varepsilon - 2x}{2\varepsilon}}$$

Example: Analysis

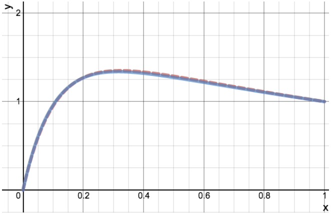

Figure 3: Exact solution (red) and composite approximation (blue) at $\varepsilon = 0.2$

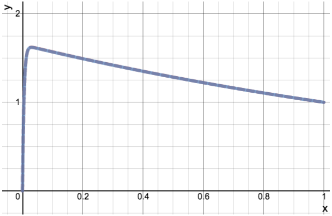

Figure 4: Exact solution (red) and composite approximation (blue) at $\varepsilon = 0.01$

Conclusion

To recap:

- Singular perturbation
- Boundary layer problems
- Method of matched asymptotic expansions
- Applications
    - Navier-Stokes Equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \varepsilon \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\varepsilon = \frac{1}{Re}$$

    - Boundary layer theory

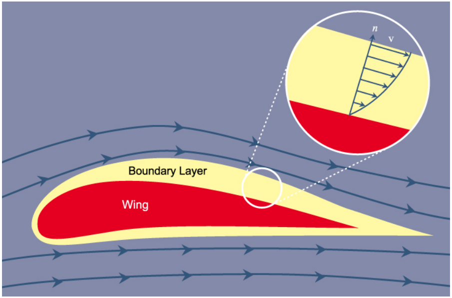

Figure 5: Boundary layer flow

Thank You

End

References

- Boundary layer. Digital image. *How Things Fly.* Smithsonian National Air and Space Museum, n.d. Web. 25 Feb. 2016.
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