Group Theory in Physics

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Motivation

Group theory is an important subject in theoretical physics, with a wide variety of applications, from particle physics to electricity and magnetism as it allows for the exploitation of symmetries to find solutions to difficult problems. Groups allow for representations of the underlying symmetries and provides tools for us to mathematically characterize solutions solely based on these symmetries.

Group

Group A set G under a closed binary operation '$\cdot$' called the group operation, forms a **group** if $\forall a, b, c \in G$ the operation satisfies the following conditions:

1. Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
2. Identity Element: $\exists$ an element $e \in G$, such that $a \cdot e = e \cdot a = a$. This element is known as the identity element.
3. Inverse: $\forall a \in G \; \exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$

A group where the group operation is commutative is then known as an **abelian** group.
A subgroup of a group G is a set $V \subseteq G$ such that it is a group under the group operation inherited from G.

Groups

Group homomorphism For two groups, G_1 and G_2, we can define a mapping known as a group homomorphism. This mapping preserves the group operation, so a mapping $s : G_1 \to G_2$ is a homomorphism if $\forall a, b \in G_1$, $s(a \cdot b) = s(a) \cdot s(b)$.
From this definition, it then follows that a homomorphism preserves inverses and maps the identity element of G_1 to the identity element of G_2.

Vector Spaces

Linear Vector Space A **linear vector space** is a set $V = \{$a, b, c, $\dots\}$ over a scalar field, $A = \{\ \alpha,\ \beta,\ \gamma,\ \dots\}$ on which two operations: addition between vectors '$+$' and multiplication '$\cdot$' by a scalar, are defined with the following properties:

1. V is an abelian group under the addition operation
2. (Closure under scalar multiplication) If $x \in V$ and $\alpha \in A$, then $\alpha \cdot x \in V$
3. (Existence of identity scalar)There exists a scalar $1 \in A$ such that for all $x \in V$, $1 \cdot x = x$. This scalar called the unity.
4. (Associativity of scalar multiplication)Multiplication by a scalar is associative, i.e for $\alpha, \beta \in A$ and $x \in V$, $\alpha(\beta x) = (\alpha\beta)x$
5. (Distributive properties) a vector space also satisfies the distributive properties, i.e. for $\alpha, \beta \in A$ and $x, y \in V$, $\alpha(x + y) = \alpha x + \beta x$ and $(\alpha + \beta)x = \alpha x + \beta x$

Linear Transformations

A **linear transformation**, A, is a mapping of elements between two linear vector spaces V and V' such that

1. for $x \in V$, $Ax \in V'$
2. for $x, y \in V$ and scalars α, β, $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$

These can also be known as a linear operator.
Some examples of linear operators are differential operators, rotation matrices, and multiplication of a $n \times n$ matrix by column vector. We say that a linear operator is defined on a vector space, V, if it maps vectors from V to V.

Matrix Representation of Linear Operators

If A is a linear transform defined on the vector space V, and the set $\{e_1, e_2, ..., e_n\}$ is the basis of V, we can construct the matrix representation by defining the components of the matrix in terms of the linear transforms on the basis vectors of V. Thus, for the tranform A, the matrix representation of the transform is given by

$$Ae_i = \sum_j (A)_{ij} e_j$$

Group Representations

Definition

(Group Representation) If there exists a homomorphism, f, from a group G to a group of linear transforms U defined on a vector space V, f can be said to be a **representation** *of the group. The dimension of the representation is the dimension of the vector space V, and the representation is said to be* **faithful** *if the homomorphism is also one-to-one. Otherwise, the representation is known as a* **degenerate** *representation.*

If the vector space V that a group representation if is defined on is an inner product space, and all linear operators of the representation are unitary, we then call that representation a **unitary representation**.

The dihedral group, D_3

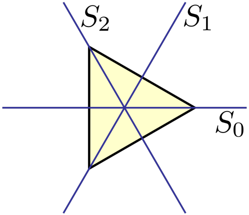

Figure : This figures show the possible symmetric reflections of an equilateral triangle.

- A group of order 6 with elements $R_0, R_1, R_2, S_0, S_1, S_2$
- R_0, R_1, R_2 represent rotations of 0°, 120°, and 240°.
- S_1, S_2, S_3 represent reflections about the axes shown above.

Constructing Group Representations

We can construct a trivial, one-dimensional representation using the map $\forall a \in D_3, f(a) = 1$.

Thus, $\forall g \in D_3,\ f(g) = 1$. Then, $\forall g_1, g_2 \in D_3$, $f(g_1)f(g_2) = 1 \cdot 1 = 1 = f(g_1g_2)$ since all elements in D_3 map to 1.

Then, f is a homomorphism and it follows that f is a representation of D_3.

However, such a representation is not very useful!

Constructing Group Representations

A non-trivial group representation of D_3 can be constructed on the three-dimensional Euclidian space, $\mathbb{R}^3$, with the standard basis vectors e_1, e_2, and e_3. Let us construct a mapping U such that the rotation elements map to their respective rotation linear transforms in three dimensions , so $U(R_0) = R_z(0^\circ)$, $U(R_1) = R_z(120^\circ)$, and $U(R_2) = R_z(240^\circ)$. The reflections map similarly to transformations in a two-dimensional vector space.

Constructing Group Representations

From these mappings, we can than construct the matrix representations of these linear operators to obtain the matrix representation of our group representations.

$$\begin{pmatrix} (U(g)e_1)_1 & (U(g)e_2)_1 & (U(g)e_3)_1 \\ (U(g)e_1)_2 & (U(g)e_2)_2 & (U(g)e_3)_2 \\ (U(g)e_1)_3 & (U(g)e_2)_3 & (U(g)e_3)_3 \end{pmatrix} = \begin{pmatrix} U(g)e_1 & U(g)e_2 & U(g)e_3 \end{pmatrix}$$

where for a vector v, v_i gives the *ith* component of v. Thus, we can than list the matrix representations of our group representation.

Constructing Group Representations

For the rotation operations:

$$U(R_0) = R_z(0^\circ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U(R_1) = R_z(120^\circ) = \begin{pmatrix} cos(120^\circ) & -sin(120^\circ) & 0 \\ sin(120^\circ) & cos(120^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 9 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U(R_2) = R_z(240^\circ) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Constructing Group Representations

For the reflection operations, through geometry and Figure 1, we can see that $S_0 e_1 = e_1$ and $S_0 e_2 = -e_2$ since e_1 is in the x-direction, and e_2 is in the y-direction. Since we are only acting in the xy plane, the z-direction is unaltered and $S_0 e_3 = e_3$

$$U(S_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U(S_1) = U(S_0 R_1) = U(S_0)U(R_1) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U(S_2) = U(S_0 R_2) = U(S_0)U(R_2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Group Representations

Furthermore, if a group, G, with a representation $U(G)$ and a normal subgroup H we can define a theorem about the nature of the representations of G/H.

Theorem

If a the group G has a non-trivial normal subgroup H, then the representation of the quotient group G/H is also a representation of G. This representation is a degenerate representation of G. In addition the converse is true. If the representation $U(G)$ of G is degenerate representation, then G has at least one non-trivial normal subgroup, H, such that $U(G)$ is a faithful representation of G/H.

Equivalent Representations

Definition

(Equivalent Representations) Two representations of a group G related by a similarity transform are said to be **equivalent***.*

Note: Two representations U and U' are related by a similarity transform if $U'(G) = SU(G)S^{-1} = \{S^{-1}U(g))S : U(g) \in U(G)\}$.

Equivalent Representations

Theorem

Any representation of a finite group defined on an inner product space has an equivalent unitary representation.

Proof.

Let $U : G \to U(G)$ be a representation of the finite group, G, defined on the inner product space V.
We construct a new invertible linear operator, $S = \sum_{g \in G} U(g)$.
Then, $\forall g \in G, \forall x, y \in V, (SU(g)S^{-1}x, SU(g)S^{-1}y)$
$= \sum_{g' \in G}(U(g')U(g)S^{-1}x, U(g')U(g)S^{-1}y)$
$= \sum_{g'' \in G}(U(g'')S^{-1}x, U(g'')S^{-1}y) = (SS^{-1}x, SS^{-1}y) = (x, y)$
Thus, $D(G) = SU(G)S^{-1}$ is an equivalent unitary representation of G. $\square$

Ex: Equivalent Representations

$$R_x(90^\circ)R_z(0^\circ)R_x(90^\circ)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_y(0^\circ)$$

$$R_x(90^\circ)R_z(120^\circ)R_x(90^\circ)^{-1} = \begin{pmatrix} -1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{pmatrix} = R_y(120^\circ)$$

$$R_x(90^\circ)R_z(240^\circ)R_x(90^\circ)^{-1} = \begin{pmatrix} -1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{pmatrix} = R_y(240^\circ)$$

Ex: Equivalent Representations

$$M(R_x(90^\circ))U(S_0)M(R_x(90^\circ))^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$M(R_x(90^\circ))U(S_1)M(R_x(90^\circ))^{-1} = \begin{pmatrix} -1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{pmatrix}$$

$$M(R_x(90^\circ))U(S_1)M(R_x(90^\circ))^{-1} = \begin{pmatrix} -1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{pmatrix}$$

Irreducibility

Consider a group representation $U : G \to U(G)$ of the group G defined on a vector space V with dimension n. We the can be block-diagonalize matrix representations of $U(G)$ to take the form

$$U(g) = \begin{pmatrix} U_1(g) & \mathbf{0} \\ \mathbf{0} & U_2(U(g)) \end{pmatrix} \forall g \in G$$

$U(g)$ the **direct sum** of matrices $U_1(g)$ and $U_2(g)$. The group $U(G)$ is then the **direct product** of the group $U_1(G)$ and $U_2(G)$ and the representation U is a direct product of the representations U_1 and U_2. We can write this as $U(G) = U_1(G) \times U_2(G)$. Each of these matrix representations operates on a vector subspace of V, V_1 and V_2.

Invariant Subspaces

Definition

(Invariant Subspaces) Let $U : G \to U(G)$ *be a representation of the group* G *defined a n-dimensional vector space* V*. We then say that a subspace of* V*,* V' *is an* **invariant subspace** *with respect to* $U(G)$ *if* $\forall x \in V', \& \forall g \in G$*,* $U(g)x \in V'$*. Thus,* V' *is closed under the linear operators of the group representation.*

We can then say an invariant subspace is **minimal** if it contains no non-trivial invariant subspaces.

Irreducible Representations

Definition

(Irreducible Representation) A representation $U : G \to U(G)$ *defined on a vector space* V *is* **irreducible** *if there is no non-trivial invariant subspace with respect to* $U(G)$*. Otherwise, we call* $U(G)$ **reducible**. *If the orthogonal complement to* V'*,* $V'' = \{x : x \in V | \forall v \in V', \ (x, v) = 0\}$ *is also an invariant subspace, then* $U(G)$ *is* **fully reducible**

Irreducible Representations

Theorem

If a unitary representation is reducible, then it is fully reducible.

Proof.

Let $U(G)$ be a unitary reducible representation of a finite group G defined on the inner product space V. Then V contains a proper subspace V_1 that is invariant with respect to $U(G)$. Call the dimension of V m and the dimension of V_1 n. Let us choose an orthogonal basis $\{e_1, \ldots, e_m\}$ for V such that $e_1, \ldots, e_n$ spans V_1. The space that is spanned by $\{e_{n+1}, \ldots, e_m\}$ is V_2. Since V_1 is invariant, for $i = 1, \ldots, m$ and $\forall g \in G$, $U(g)e_i \in V_1$. For $i = 1, \ldots, n$, $j = n+1, \ldots, m$ $g \in G$, $(e_i, e_j) = (e_i, U(g)^{-1}U(g)e_j) = (U(g)e_i, U(g)e_j) = 0$. Then $U(g)e_j$ is in V_2 since it is orthogonal to all elements of V_1. Thus, V_2 is an invariant subspace and an orthogonal complement to V_1.

Example of Reducible Representations

Take the our 3-dimensional representation of D_3. It has two invariant orthogonal subgroups, the space spanned by $\{e_1, e_2\}$, and the space spanned by $\{e_3\}$, i.e. the xy plane and the z-axis.

The 3-dimensional representation is then a direct product of the trivial representation discussed earlier, which acts on the z -axis space, and a two dimension representation consisting of the rotation and reflection operators in the plane.

$$U(g) = \begin{pmatrix} U_1(g) & \mathbf{0} \\ \mathbf{0} & U_2(U(g)) \end{pmatrix} = \forall g \in G$$

Character of a Representation

Definition

(Character of a Group Representation) Given a representation $U : G \to U(G)$ of the group G, the **character** $\chi(g)$ *the trace of the matrix representation of $U(g)$.*

Then, we can identify equivalent representations by calculating the characters of the group elements. If the characters are the same, then the group representations are equivalent. Furthermore, all elements of a conjugacy class have the same character.

Properties of Irreducible, Inequivalent Representation

Theorem

$$\frac{n_G}{n_\mu}\sum_{g\in G} D_\mu(g)_{ik} D_\nu(g)_{lj} = \delta_{\mu\nu}\delta_{il}\delta_{kj}$$

- Every representation of a group G can be written as a direct product of irreducible representations.
- The number of irreducible representations of a group is equivalent to the number of conjugacy classes of a group.
- For a given group representation U of a group G with order n, $n = \sum_G \chi_U(G)$.

Applications of Representation Theory in Physics

In quantum mechanics, a system with n-degrees of freedom is completely described by a complex-valued function called a wave function $\Psi(\mathbf{r}, t)$, where $\mathbf{r}$ is a n-dimensional vector describing those degrees of freedom, and t is time.

These wave functions describing the system are found by solving the famous Schrödinger equation,

$$H(\mathbf{r}, t)\Psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t)$$

where H is an Hermitian operator known as the Hamiltonian. However, we will focus on the time independent version of this equation

$$H(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

Applications of Representation Theory in Physics

We again consider a time-independent Hamiltonian, $H(\mathbf{r})$, an arbitrary time-independent wave function $\psi(\mathbf{r})$, the vector space V such that $\mathbf{r} \in V$, and a set of coordinate transforms G defined on V. These coordinate transforms then induce a group of transformations, $T(G)$, in the space of wave functions. We then define the transforms $\forall g \in G$

$$T(g)\psi(\mathbf{r}) = \psi'(\mathbf{r}) = \psi(g^{-1}\mathbf{r}) \tag{1}$$

if the Hamiltonian is invariant under the similarity transforms from all elements of $T(G)$, i.e. if $\forall T(g) \in T(G), T(g)HT^{-1}(g) = H$, then the group G is a **symmetry group of the Hamiltonian** and T is a representation of G.

Applications of Representation Theory in Physics

Two properties arise out of the existence of a symmetry group, G, for the Hamiltonian.

1. The eigenfunctions and eigenvalues of H may be labeled by the irreducible representations, $T^{\mu}(G)$ of the symmetry group, G, such that we write them E^{μ} and ψ^{μ}.
2. The state, E^{μ} corresponding to the μ-representation of G will at least have a degeneracy on order of, n, dimensionality of the representation. We say that degeneracy of a eigenvalue implies that there are multiple eigenfunctions that share the eigenvalue. A degeneracy on the order of n, means that there are n eigenfunctions that share the same eigenvalue.

Applications of Representation Theory in Physics

Consider a Hamiltonian with the symmetry group, C_3. C_3 is a subgroup of the dihedral group, D_3 containing only the three rotation elements, R_0, R_1, R_2. In this case this represents 120° rotations about the z-axis

$T^1(C_3)$ is the trivial representation, mapping all elements to the identity operator. $T^2(C_3)$ maps R_1 to $e^{2\pi i/3}$ and $R2$ to $e^{2\pi i/3}$ while R_0 maps to 1. $T^3(C_3)$ maps R_1 to $e^{4\pi i/3}$, R_2 to $e^{8\pi i/3}$, and R_0 again to 1.

$$T^{\alpha}(R_1) = e^{2\pi(\alpha-1)i/3}, T^{\alpha}(R_2) = e^{4\pi(\alpha-1)i/3}, T^{\alpha}(R_0) = 1 \quad (2)$$

Since the dimensionality of all the group representations is 1, then we can immediately assume that there are no degenerate eigenstates. There will be three eigenstates, each labeled by a representation of of C_3.

Applications of Representation Theory in Physics

For a eigenfunction labeled ϕ^α,

$$\phi^\alpha(\theta - 2\pi/3) = e^{2\pi(\alpha-1)i/3}\phi^\alpha(\theta) \tag{3}$$

by definition of the of spacial transform we defined earlier. From this we can make an *ansatz* that the eigenfunctions for H take on the form

$$\phi^\alpha(\theta) = u_\alpha(\theta)e^{i(\alpha-1)\theta} \tag{4}$$

Plugging equation 4 into equation 3, we get that

$$u_\alpha(\theta - 2\pi/3)e^{i(\alpha-1)(\theta-2\pi/3)} = e^{2\pi(\alpha-1)i/3}u_\alpha(\theta)e^{i(\alpha-1)\theta} \tag{5}$$

which simplifies to $u_\alpha(\theta - 2\pi/3) = u_\alpha(\theta)$, verifying that our solutions have periodic symmetry for θ and that we must only solve in the range $[0, 2\pi/3)$.

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