Applied Mathematics

Math 395 Spring 2009 — Fowler 301 Tue 3:00pm - 4:25pm

 — http://faculty.oxy.edu/ron/math/395/09/

Class 4: Tuesday February 10

TITLE Scaling or Non-Dimensionalization, Part 2
CURRENT READING Logan, Section 1.2

SUMMARY
This week we will continue looking at scaling problems by examining the rather complicated (but famous) projectile problem.

The Projectile Problem

Let us analyze the motion of a projectile thrust into the atmosphere from the surface of the earth vertically. Turns out the governing equation is Newton's second law of motion, $F = ma$ which looks like

$$m\frac{d^2h}{dt^2} = -G\frac{Mm}{(h+R)^2} \tag{1}$$

Where R is the radius of the earth, M is its mass, m is the mass of the projectile, V its velocity and h its height. On the earth's surface, i.e. at $h = 0$ the weight of the projectile is equal to the gravitational force, so $mg = \frac{GMm}{R^2}$ or $g = \frac{GM}{R^2}$.

Thus (1) becomes

$$\frac{d^2h}{dt^2} = -\frac{R^2 g}{(h+R)^2} \tag{2}$$

with initial conditions

$$h(0) = 0 \quad h'(0) = V \tag{3}$$

These ODE in (2) plus the equation in (3) represents an initial value problem which is the mathematical model for this projectile problem.

The book (Logan 27) does a non-dimensional analysis of the variables involved:

$[t]$= time T

$[h]$= length L

$[R]$ = length L

$[V]$= velocity LT^-1

$[g]$= acceleration LT^{-2}

It turns out that there are three dimensionless quantities

$$\pi_1 = \frac{h}{R} \qquad \pi_2 = \frac{t}{R/V} \qquad \pi_3 = \frac{V}{\sqrt{gR}}$$

which implies $\pi_1 = f(\pi_2, \pi_3)$ or

$$\frac{h}{R} = f\left(\frac{t}{R/V}, \frac{V}{\sqrt{gR}}\right)$$

This above formula is interesting because if you differentiate both sides with respect to t one can find that the time to the maximum value of h depends only on the quantity $\pi_3 = \frac{V}{\sqrt{gR}}$ which is an important result.

Scaling The Projectile Problem

We need to choose a characterisic time t_c and a characteristic length h_c, where $\tilde{t} = \dfrac{t}{t_c}$ and $\tilde{h} = \dfrac{h}{h_c}$

Interestingly, there are three choices of pairs for t_c and h_c which are physically meaningful. Let's look at the choices and see how they change the IVP rpresenting the model given in (2) and (3).

CHOICE 1

$$\tilde{t} = \frac{t}{R/V}, \tilde{h} = \frac{h}{R} \tag{4}$$

CHOICE 2

$$\tilde{t} = \frac{t}{\sqrt{R/g}}, \tilde{h} = \frac{h}{R} \tag{5}$$

CHOICE 3

$$\tilde{t} = \frac{t}{V/g}, \tilde{h} = \frac{h}{V^2/g} \tag{6}$$

EXAMPLE

Let's use each of these choices to demonstrate what happens to the IVP for the projectile model.

CHOICE 1

$\tilde{t} = \dfrac{t}{R/V}, \tilde{h} = \dfrac{h}{R}$

The IVP becomes

$$\epsilon \frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{h})^2}, \qquad \tilde{h}(0) = 0, \quad \frac{d\tilde{h}}{d\tilde{t}} = 1 \tag{7}$$

CHOICE 2

$\tilde{t} = \dfrac{t}{\sqrt{R/g}}, \tilde{h} = \dfrac{h}{R}$

The IVP becomes

$$\frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{h})^2}, \qquad \tilde{h}(0) = 0, \quad \frac{d\tilde{h}}{d\tilde{t}} = \sqrt{\epsilon} \tag{8}$$

CHOICE 3

$\tilde{t} = \dfrac{t}{V/g}, \tilde{h} = \dfrac{h}{V^2/g}$

The IVP becomes

$$\frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\epsilon\tilde{h})^2}, \qquad \tilde{h}(0) = 0, \quad \frac{d\tilde{h}}{d\tilde{t}} = 1 \tag{9}$$