Applied Mathematics

Math 395 Spring 2009 | Fowler 301 Tue 3:00pm - 4:25pm

 | http://faculty.oxy.edu/ron/math/395/09/

Class 11: Tuesday April 21

TITLE Uniform Solutions and Asymptotic Matching
CURRENT READING Logan, Sections 2.2.2 and 2.2.3

SUMMARY
This week we will learn how to do asymptotic matching in order to obtain an exact solution to an ODE with a boundary layer that is valid inside and outside of the layer.

CONSIDER
Given the boundary value problem

$$\epsilon\frac{d^2y}{dx^2} + (1+\epsilon)\frac{dy}{dx} + y = 0, \text{ where } \epsilon \ll 1 \text{ and } \quad 0 < x < 1 \text{ with } y(0) = 0, \quad y(1) = 1. \tag{1}$$

We have solved the outer problem for y_{outer} which is valid when ϵ is ignored, i.e. in the range where $\mathcal{O}(\epsilon) < x \leq 1$, so you can let $\epsilon = 0$ in the original problem given in (1). In that case the problem becomes

$$y'_{outer} + y_{outer} = 0, \quad y_{outer}(1) = 1 \tag{2}$$

Solving the IVP in (2) gives us the solution $y_{outer}(x) = e^{1-x}$.

However, in the boundary layer (inner solution) we have to do more work. We will rescale the independent variable x using the following:

$$\xi = \frac{x}{\delta(\epsilon)} \text{ and } y(\xi) = y(x) = y(\xi\delta(\epsilon) \tag{3}$$

and plug in the new variables in (3) into the original equation in (1) produces

$$\frac{\epsilon}{\delta(\epsilon)^2}\frac{d^2Y}{d\xi^2} + \frac{(1+\epsilon)}{\delta(\epsilon)}\frac{dY}{d\xi} + Y(\xi) = 0 \tag{4}$$

There are four terms to do a dominant balancing of, $\frac{\epsilon}{\delta(\epsilon)^2}$, $\frac{1}{\delta(\epsilon)}$, $\frac{\epsilon}{\delta(\epsilon)}$ and 1.

EXAMPLE

Let's show that a consistent balancing only is available if $\delta(\epsilon) = \epsilon$ is chosen.

Choosing the scaling $\delta(\epsilon) = \epsilon$ and plugging back into (4) leads to

$$\epsilon Y'' + Y' + \epsilon Y' + \epsilon Y = 0 \tag{5}$$

which is an ODE that can e approximated using regular perturbation, so we set $\epsilon = 0$ and consider the leading order problem $Y'' + Y' = 0$, which has the solution

$Y(\xi) = A + Be^{-\xi}$ but since this is the inner solution, it should satisfy the inner boundary condition of $y = 0$ at $x = 0$ which means that $Y = 0$ when $\xi = 0$ so that $B = -A$ and the inner solution has the form $Y(\xi) = A(1 - e^{-\xi})$. If we want to convert back into x variables from ξ we know that $\xi = \dfrac{x}{\epsilon}$

To summarize, we know have

$$\begin{aligned} y_{inner}(x) &= A(1 - e^{-x/\epsilon}), \text{ when } 0 \le x \le \mathcal{O}(\epsilon) \\ y_{outer}(x) &= e^{1-x}, \text{ when } \mathcal{O}(\epsilon) < x \le 1 \end{aligned}$$

The process of finding the value of the constant involves asymptotic matching.

Asymptotic Matching

In order to find the unknown constant in the inner solution we need a matching condition. It turns out that this is

$$\lim_{x \to 0^+} y_{outer}(x) = \lim_{\xi \to \infty} y_{inner}(\xi) = M \tag{6}$$

where M is the matched value equal to the value of both limits.

If we do this for our problem above, we will see that $M = e$.

Exercise

Use the matching condition $\lim_{x \to 0^+} y_{outer}(x) = \lim_{\xi \to \infty} Y(\xi)$ to confirm the value for the unknown constant in y_{inner}.

Uniform Expansion

To find a uniform expansion which is valid for the entire domain of interest (from $0 \leq x \leq 1$) instead of a piecewise defined function, we obtain $y_{uniform}(x)$ by adding together the inner and outer solutions and subtracting the common term, so

$$y_{uniform}(x) = y_{inner}(x) + y_{outer} - M \tag{7}$$

Thus $y_{uniform}(x) = e^{1-x} + e(1 - e^{-x/\epsilon}) - e = e^{1-x} - e^{1-x/\epsilon}$ is the function which satisfies (1) to leading order, in other words, as $\epsilon \to 0^+$.

EXAMPLE

Let's show that our uniform solution $y_u(x)$ satisfies the BVP and the ODE.

$$\epsilon \frac{d^2 y_u}{dx^2} + (1+\epsilon)\frac{dy_u}{dx} + y = 0, \qquad y_u(0) = 0, \quad y_u(1) = 1.$$

GROUPWORK

Let's try to come up with a unform expansion for the solution to

$$\epsilon y'' + y' = 2x, \quad y(0) = 1, \quad y(1) = 1, \qquad 0 < x < 1, 0 < \epsilon \ll 1 \tag{8}$$

BONUS Homework Questions for Math 395: Applied Mathematics due TUE APR 28

For each of the problems, use singular perturbation methods to obtain a uniform approximate solutio to the following boundary value problems. Assume $0 < \epsilon \ll 1$ and $0 < x < 1$.

GROUP 1: Logan, page 121, Question 1(a)
(a) $\epsilon y'' + 2y' + y = 0 \quad y(0) = 0, \quad y(1) = 1$

GROUP 2: Logan, page 121, Question 1(b)
(b) $\epsilon y'' + y' + y^2 = 0, \quad y(0) = 1/4, \quad y(1) = 1/2$

GROUP 3: Logan, page 121, Question 1(c)
(c) $\epsilon y'' + (1 + x)y' = 1, \quad y(0) = 0, \quad y(1) = 1 + \ln(2)$