Numerical Analysis

Math 370 Spring 2009
MWF 11:30am - 12:25pm Fowler 110

`http://faculty.oxy.edu/ron/math/370/09/`

Worksheet 8

SUMMARY Introduction to Root Finding
READING Recktenwald, 6.1.1 (240-250); Mathews & Fink 51-62

| EXAMPLE |

Consider a ball constructed of wood which has a density of $\rho = 0.638$ grams per cubic cm and the radius is $r = 10$ cm. How much of the ball will be submerged when it is in water (with unit density)? Let x be the current depth of the sphere. The radius of the amount of the spherical section under water is obtained using Pythagoras' theorem with $r - x$ and r

$M_w =$ Mass of water displaced $= 1 \cdot \int_0^d \pi(r^2 - (r-x)^2)dx = \frac{\pi}{3}d^2(3r - d)$

$M_b =$ Mass of ball $= 4\pi r^3 \rho/3$

What's the equation which must be solved to find d, the distance below the surface the ball will float? (Produce an equation for d of the form $f(d) = 0$ with d being the only letter present. You will need to use the values of ρ and r.)

Question
How would you solve this equation for d?

Root-Finding

We will be looking at algorithms for the solution of equations of one variable, i.e. equations of the form $f(x) = 0$. This is often referred to as finding the **roots** of the equation $f(x) = 0$ or finding the **zeroes** of the function $f(x)$.

Bracketing The Root

How do we know where the roots of a function $f(x)$ are? How can we "bracket" a zero of $f(x)$? [HINT: recall the Intermediate Value Theorem!]

GROUPWORK

The MATLAB function **brackplot** will do this for us. Go to the computers and run `brackplot` on the function you need to find zeroes of to find d. I have made a function called **sphere.m** which you can use to help you. What do you see? How many roots are there? What range did you ask `brackplot` to search on?

The Bisection Method of Bolzano

The bisection algorithm produces a sequence of approximations $\{p_n\}$ to the zero of the function $f(x)$

where $p_n = a_n + \frac{b_n - a_n}{2} = \frac{a_n + b_n}{2}$ and the n-th bracket is described by $[a_n, b_n]$

Write down the Bisection Algorithm in pseudocode here:

bisect.m

In the NMM Toolbox, we have an implementation of the bisection algorithm in **bisect.m**. Use MATLAB to find the value of d which we have been looking for which tells us how much of the pine sphere is submerged.

$d =$

General Root-Finding Algorithm

```
1. Plot the function, in order to get an initial guess for the root
 and to check for problems
2. Select an initial guess [or bracket ]
3. Iteratively refine your initial guess
4. Decide you are "converged" (If NOT, Go To 3.)
5. Stop
```

demobisect.m

There is another implementation of the Bisection Algorithm in a file called `bisect.m` found in **S:/Math Courses/Math 370/Spring2009/mathews** .
Modify this m-file to find the root of $f(d) = 2552 - 30d^2 + d^3$
How many steps does it take to converge? Using what initial bracket?

Analyzing Convergence of Bisection

Write down an expression for the size of $|b_n - a_n|$ which depends on $b - a$ and the n-th iterate (note: $|b_0 - a_0| = b - a$)

Solve this formula for n.

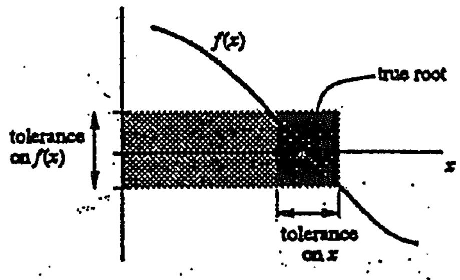
Try and predict how many iterations it will take Bisection to find the zero of $f(x) = \log(x) - 5 + x$ on the interval [1,9] to 5 decimal places

Go to the computer and see how many iterations **demobisect.m** actually takes to converge. Explain.

Convergence Criteria

There are a number of different ways to consider that a method has "converged"
There is convergence criteria on $f(x)$ and convergence criteria on x

Convergence Criteria on x



x_k = current guess at the root

x_{k-1} = previous guess at the root

Absolute tolerance: $|x_k - x_{k-1}| < \delta_x$

Relative tolerance: $\left|\dfrac{x_k - x_{k-1}}{b-a}\right| < \delta_x$

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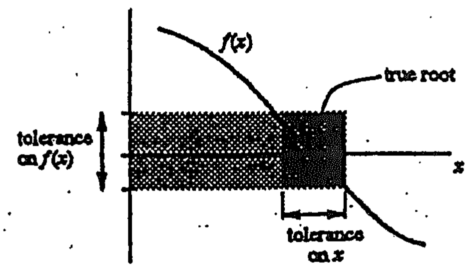
Convergence Criteria on $f(x)$



Absolute tolerance: $|f(x_k)| < \delta_f$

Relative tolerance:

$$|f(x_k)| < \delta_f \max\{|f(a_0)|, |f(b_0)|\}$$

where a_0 and b_0 are the original brackets

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Question

There is also relative convergence versus absolute convergence. Which do you think is the "best" method of assessing convergence?