Numerical Analysis

Math 370 Spring 2009 MWF 11:30am - 12:25pm Fowler 110

 http://faculty.oxy.edu/ron/math/370/09/

Class 3

SUMMARY Sequences, Order and "big Oh"
CURRENT READING Mathews & Fink (Sec 1.3)

Convergence of a Sequence
In a large number of numerical problems we will get a sequence of approximate answers to the single real number which is the exact solution of the problem we are looking at (e.g., a definite integral, a particular value of a solution to an initial value problem, a root of a function, etc.) We often write this sequence as $x_1, x_2, x_3, \ldots$ and the limit as L or x_∞ and denote this by

$$\lim_{n \to \infty} x_n = L$$

Can you recall the formal definition of the above limit of a sequence?

In English, write down what the definition means to you, **in your own words**.

Draw a picture representing this definition:

As we solve problems numerically, we often generate a sequence of approximations x_1, x_2, x_3 which approach an exact answer x_∞.
We are interested in looking at **rate of convergence** of sequences. Often we want to compare how fast one sequence is converging to its limit relative to another convergent sequence. This is a convenient way of describing and evaluating solution algorithms.

Definition

Suppose we know that a sequence $\{\beta_n\}$ converges to β and $\{\alpha_n\}$ converges to α.
The sequence $\{\alpha_n\}$ is said to converge to α at the (**order of convergence** or) **rate of convergence** $\mathcal{O}(\beta_n)$ if there exists a positive constant K such that

$$|\alpha_n - \alpha| \leq K|\beta_n - \beta|, \qquad \text{for large } n$$

Another way of thinking of this is to say that

$$\lim_{n\to\infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|} = K, \qquad \text{where } 0 < K < \infty$$

This is often written as $\alpha_n = \alpha + \mathcal{O}(\beta_n)$

We read this (in English) as :

Similarly, we say that $\{\alpha_n\}$ is $o(\beta_n)$ if

$$\lim_{n\to\infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|} = 0$$

and that $\{\alpha_n\}$ is **equivalent** to $\{\beta_n\}$ (this is written $\alpha_n \sim \beta_n$) if

$$\lim_{n\to\infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|} = 1$$

For most practical purposes, the limiting values α and β are zero and the β_n sequence we deal with most often has the form $1/n^p$.

Key Idea

The main point to remember is that "big Oh" is about taking one sequence, say α_n and bounding it by another, say β_n as n gets larger and larger. In that case, $\{\alpha_n\}$ is $\mathcal{O}(\beta_n)$ if there exist constants K and N such that

$$|\alpha_n| \leq K|\beta_n| \text{ for all } n \geq N$$

You can think of it as a race, and $\alpha_n = \mathcal{O}(\beta_n)$ is saying that α_n is running no faster than β_n. $\alpha_n = o(\beta_n)$ means that α_n is **faster** than β_n. α_n is approaching its limit faster than β_n is approaching its.

EXAMPLE

Show that $x_n = \dfrac{n+1}{n^2}$ is $\mathcal{O}\left(\dfrac{1}{n}\right)$ and is also $o(1)$.

What does this result tell you about the meaning and meaningfulness of saying $x_n = o(1)$?

Exercise

What is the order of convergence of $t_n = \dfrac{1}{n \ln n}$?

EXAMPLE

What is the order of convergence of $y_n = e^{-n}$?

<u>GroupWork</u>

Find the order of convergence of the following sequences as $n \to \infty$. Use $\mathcal{O}$ and o where appropriate.

1. $x_n = 5n^2 + 9n^3 + 1$

2. $x_n = e^{-n} + 5/n$

3. $x_n = \sqrt{n+3}\dfrac{1}{n^2+4}$

4. $x_n = \dfrac{1}{\ln n}$