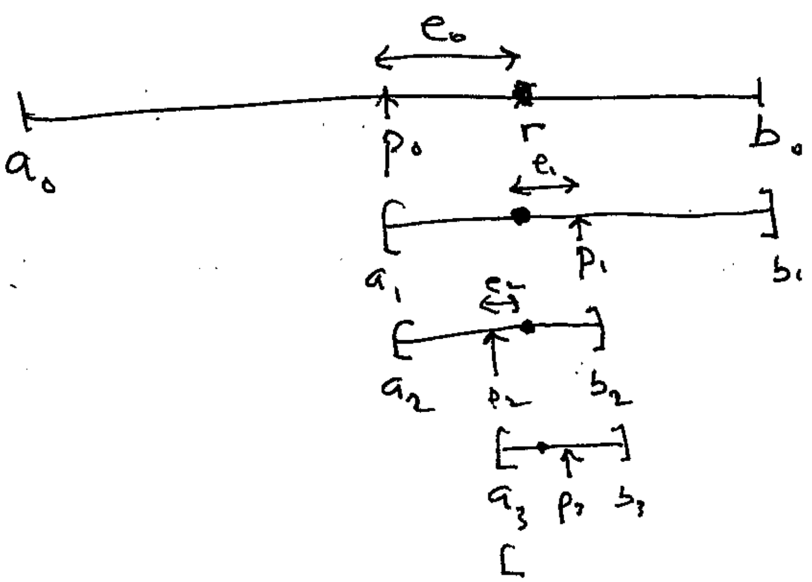
HW SET 2

Fall 2000
Math 370

1. (5pts) $p_n = \frac{1}{2}(a_n+b_n)$ $\quad r = \lim_{n\to\infty} p_n$ $\quad e_n = |r-p_n|$

This is referring to thinking of BISECTION algorithm as a method of generating a convergent sequence $\{p_n\}$.



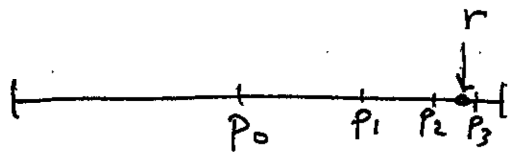
(a) Clearly, the error $|e_0| = |r-p_0| \le \frac{1}{2}(b_0-a_0)$

$$|e_1| = |r-p_1| \le \frac{1}{2}(b_1-a_1) = \frac{1}{2}\cdot\frac{1}{2}(b_0-$$

$$|e_2| = |r-p_2| \le \frac{1}{2}(b_2-a_2) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}(b_0$$

$$|e_n| \le \frac{1}{2^{n+1}}(b_0-a_0) = \frac{b_1-a_1}{2^n}$$

(b) Observe the diagrams



$$|p_1 - p_0| = b_2 - a_2$$

$$|p_2 - p_1| = b_3 - a_3$$

$$|p_n - p_{n-1}| = b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = \frac{b_0 - a_0}{2^{n+1}}$$

But in (a) we showed $|e_n| = |r - p_n| \le \frac{1}{2}(b_n - a_n)$

So, which is bigger, $|r - p_n|$ or $|p_n - p_{n-1}|$?

Clearly $|r - p_n|$ must be smaller than $\frac{b_n - a_n}{2} = |p_n - p_{n-1}|$ since r must exist on the interval ~~(a_n, b_n) and~~ $(\frac{b_n + a_n}{2}, b_n)$ or $(a_n, \frac{b_n + a_n}{2})$ which means $\left|r - \frac{b_n + a_n}{2}\right| = |r - p_n| < \frac{b_n - a_n}{2}$

The only time $|r - p_n| = \frac{b_n - a_n}{2}$ is if r is on the boundary of an interval.

(c). Use the fact from (a) $|e_n| \le 2^{-n}(b_0 - a_0)$

Thus $|e_n - 0| \le K \cdot 2^{-n}$ for all $n > N$ and some K:

From definition of Θ, $|e_n| = 0 + \Theta(2^{-n})$

$$\lim_{n\to\infty} |e_n| = \left|\lim_{n\to\infty} e_n\right| = \left|\lim_{n\to\infty} r - p_n\right| = \left|\lim_{n\to\infty} r - \lim_{n\to\infty} p_n\right|$$

$$= r - r = \boxed{0 = \lim_{n\to\infty} |e_n|}$$

(d) The $\lim_{n\to\infty} \frac{|r - p_{n+1}|}{|r - p_n|} = K$ is just the definition of linear convergence.

Another definition is $|e_{n+1}| \approx K|e_n|$

For Bisection, we know $|e_{n+1}| \approx \frac{1}{2}|e_n|$. Thus its linearly convergent.

2. (4pts)

To find $\sqrt[3]{25}$ within 10^{-4} using BISECTION

~~set~~ find the root of $f(x) = x^3 - 25$ to within 10^{-4}.

Presumably "within 10^{-4}" is referring to an XTOL of 10^{-4}.

See attached sheet

question.txt

```
type cubic.m


function f  = cubic(x)
f = x.^3-25.;
xb = [0 3]


xb =

     0     3

cubic(xb)


ans =

   -25     2

bisect('cubic',xb,.0001,5*eps,1)


Bisection iterations for cubic.m
   it       xm             fm
    1   1.5000e+000   -2.1625e+001
    2   2.2500e+000   -1.3609e+001
    3   2.6250e+000   -6.9121e+000
    4   2.8125e+000   -2.7527e+000
    5   2.9062e+000   -4.5297e-001
    6   2.9531e+000    7.5405e-001
    7   2.9297e+000    1.4571e-001
    8   2.9180e+000   -1.5483e-001
    9   2.9238e+000   -4.8632e-003
   10   2.9268e+000    7.0348e-002
   11   2.9253e+000    3.2723e-002
   12   2.9246e+000    1.3925e-002
   13   2.9242e+000    4.5299e-003
   14   2.9240e+000   -1.6692e-004
   15   2.9241e+000    2.1814e-003

ans =

    2.9241

--more--diary off
```

3 (5pts)

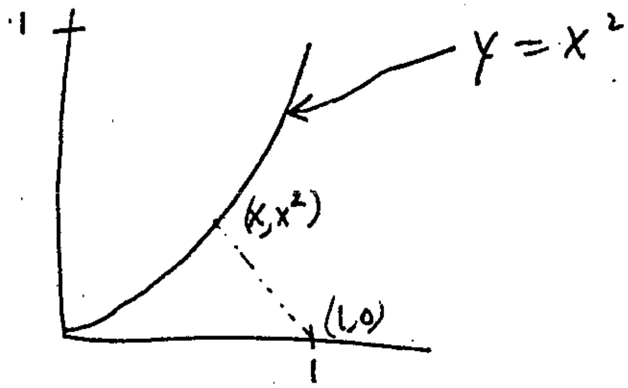
$f(x) = x^2 - 6$

Exact answer $= \sqrt{6} = 2.44949$

You could use TRUBASIC or do this by hand

BISECTION	FALSEPOS	SECANT	NEWTON
$p_0 = 2.5$	2.4000	2.4000	2.5
$p_1 = 2.25$	2.4444	2.45454	2.45
$p_2 = 2.375$	2.4480	2.449438	2.4494
$p_3 = 2.4375$	2.4494	2.449490	2.4494

Newton, obviously



$$d(x) = \sqrt{(x^2-0)^2+(x-1)^2}$$

$$d^2(x) = x^4 + (x^2-2x+1)$$

$$= x^4+x^2-2x+1$$

To minimize $(d(x))^2$, solve $[(d(x))^2]' = 0$

$$f(x) = 4x^3+2x-2 = 0$$

$$f'(x) = 12x^2+2$$

Newton's converges to $x = 0.58975$ in 4 iterations with $TOL = 1e^{-4}$ and $p_0 = 0.5$

$$x_{n+1} = x_n - \frac{4x_n^3+2x_n-2}{12x_n^2-2}$$

$$= x_n - \frac{2x_n^3+x_n-1}{6x_n^2-1}$$

$$= \frac{6x_n^3-x_n-2x_n^3-x_n+1}{6x_n^2-1}$$

$$x_{n+1} = \frac{4x_n^3+1-2x_n}{6x_n^2-1}$$

Co-ordinates of the closest point are $(0.58975, 0.34781)$

$$x + y = 20$$

$$(x+\sqrt{x})(y+\sqrt{y}) = 155.55$$

$$(x+\sqrt{x})(20-x+\sqrt{20-x}) = 155.55$$

$$f(x) = 20x + 20\sqrt{x} - x^2 - x^{3/2} + x\sqrt{20-x} + \sqrt{20x-x^2} - 155.55 =$$

Do you wat to solve this using Newton's?

No! Try SECANT! (why take a derivative)

$$f(0) = -155.55$$

$$f(20) = -155.55$$

Oops! Try a different bracket

$$f(10) = (10+\sqrt{10})(10+\sqrt{10}) - 155.55 > 0$$

Actually, running SECANT (ALG24.TRU) with $p_0 = 0$, $p_1 = 4$ converges in 6 steps with an ~~rela~~ absolute error tolerance of $1e^{-4}$

$$x = 6.5128$$

$$y = 20 - x = 13.4872$$

1 (p 54, #1)

$$f(x) = x^4 + 2x^2 - x - 3$$

when $f(p) = 0$ we need to show each of the $g(x)$ functions has a fixed point

(a) $g_1(x) = [3 + x - 2x^2]^{1/4} = x$

$$3 + x - 2x^2 = x^4$$

$$\Rightarrow x^4 - 3 - x + 2x^2 = 0$$

This is equivalent to $f(x) = 0$

(b) $g_2(x) = \left[\frac{x + 3 - x^4}{2}\right]^{1/2} = x$

$$\frac{x + 3 - x^4}{2} = x^2$$

$$x + 3 - x^4 = 2x^2$$

$$x^4 + 2x^2 - x - 3 = 0 = f(x)$$

(c) $g_3(x) = \left[\frac{x + 3}{x^2 + 2}\right]^{1/2} = x$

$$\Rightarrow \frac{x + 3}{x^2 + 2} = x^2$$

$$x + 3 = x^4 + 2x^2$$

$$x^4 + 2x^2 - x - 3 = 0$$

(d) $g_4(x) = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} = x$

$$3x^4 + 2x^2 + 3 = 4x^4 + 4x^2 - x$$

$$x^4 + 2x^2 - x - 3 = 0 = f(x)$$

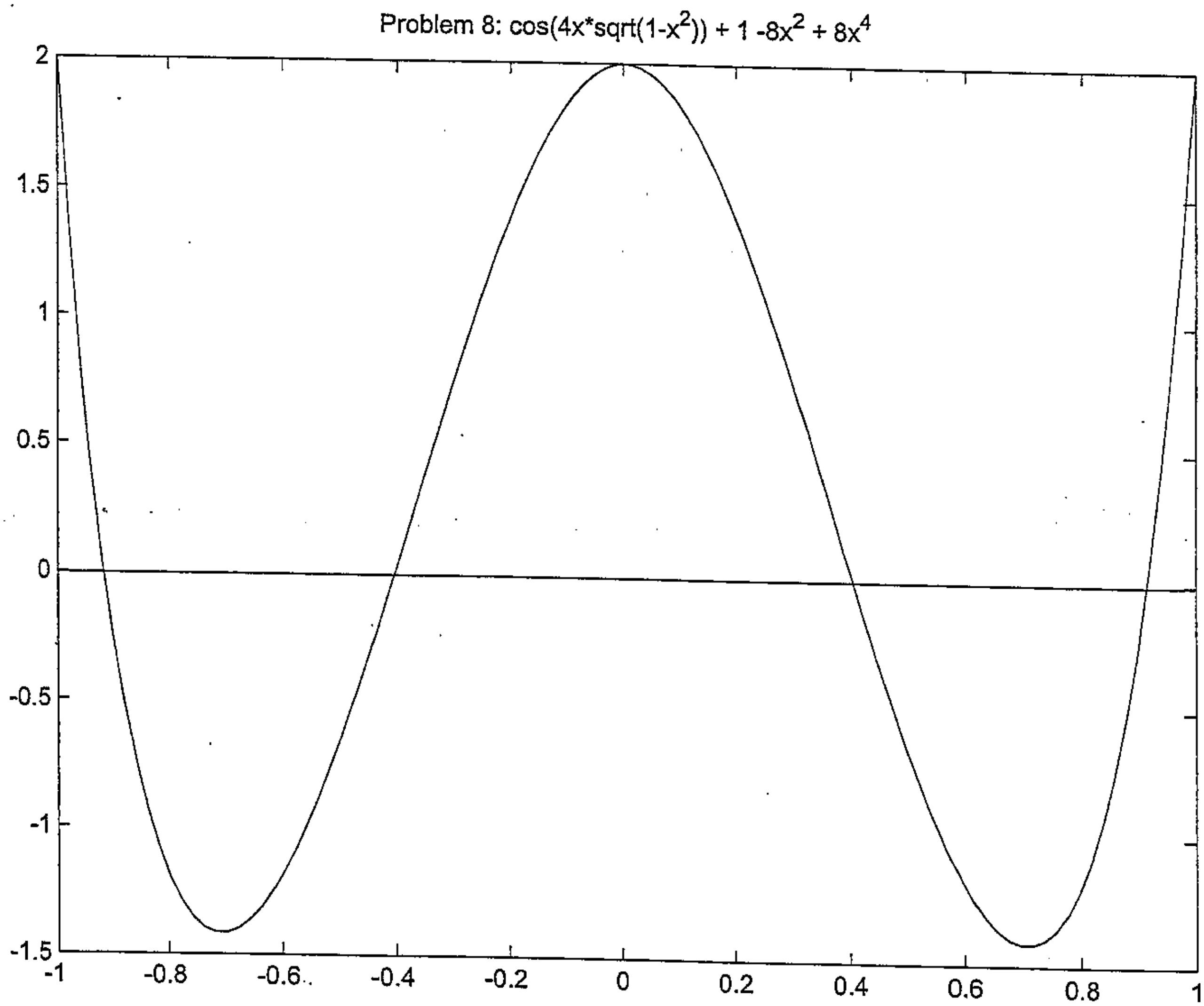
So, solving $f(x) = 0$ is equivalent to finding fixed pts for g_1, g_2, g_3 & g_4

```
type seven


function f = func(x)
%f = (3+x-2*x*x)^(1/4.);
f = ((x+3)/(x*x+2))^(1/2);
%f = ((x+3-x^4)/2)^(1/2);
%f = (3*x^4+2*x^2+3)/(4*x^3+4*x-1);

picard('seven',1,3)

  k       g(x)             x(k+1)               |x(k)-x(k+1)|
  0    1.155e+000     1.154700538379252     0.154700538379251
  1    1.116e+000     1.116427409872122     0.038273128507129
  2    1.126e+000     1.126052233002276     0.009624823130154

ans =

    1.126052233002276e+000

type seven


function f = func(x)
f = (3+x-2*x*x)^(1/4.);
%f = ((x+3)/(x*x+2))^(1/2);
%f = ((x+3-x^4)/2)^(1/2);
%f = (3*x^4+2*x^2+3)/(4*x^3+4*x-1);

picard('seven',1,3)

  k       g(x)             x(k+1)               |x(k)-x(k+1)|
  0    1.189e+000     1.189207115002721     0.189207115002721
  1    1.080e+000     1.080057752667562     0.109149362335159
  2    1.150e+000     1.149671430589383     0.069613677921820

ans =

    1.149671430589383e+000

type seven


function f = func(x)
%f = (3+x-2*x*x)^(1/4.);
%f = ((x+3)/(x*x+2))^(1/2);
f = ((x+3-x^4)/2)^(1/2);
%f = (3*x^4+2*x^2+3)/(4*x^3+4*x-1);

picard('seven',1,3)

  k       g(x)             x(k+1)               |x(k)-x(k+1)|
  0    1.225e+000     1.224744871391589     0.224744871391589
  1    9.937e-001     0.993666159077482     0.231078712314107
  2    1.229e+000     1.228568645274987     0.234902486197505

ans =

    1.228568645274987e+000

type seven
```

```
function f = func(x)
%f = (3+x-2*x*x)^(1/4.);
%f = ((x+3)/(x*x+2))^(1/2);
%f = ((x+3-x^4)/2)^(1/2);
f = (3*x^4+2*x^2+3)/(4*x^3+4*x-1);

picard('seven',1,3)

  k       g(x)            x(k+1)                |x(k)-x(k+1)|
  0     1.143e+000     1.142857142857143     0.142857142857143
  1     1.124e+000     1.124481690017895     0.018375452839247
  2     1.124e+000     1.124123163940149     0.000358526077746

ans =

    1.124123163940149e+000

diary off
```

```
type eight

function [y] = eight(x)
y = cos(4.*x.*sqrt(1-x.*x)) + 1 - 8.*x.*x + 8.*x.^4;
bisect('eight',[-1 .5],1.e-8,1e-8,0)

ans =

    0.4040

format long
bisect('eight',[-1 .5],1.e-8,1e-8,0)

ans =

   0.40397275425494

bisect('eight',[-1 -.5],1.e-8,1e-8,0)

ans =

  -0.91477101668715

bisect('eight',[.5 1],1.e-8,1e-8,0)

ans =

   0.91477101668715

bisect('eight',[0 .5],1.e-8,1e-8,0)

ans =

   0.40397275239229

bisect('eight',[0 .5],1.e-8,1e-8,1)

Bisection iterations for eight.m
   it        xm              fm
    1   2.5000e-001   1.0980e+000
    2   3.7500e-001   2.1249e-001
    3   4.3750e-001  -2.4100e-001
    4   4.0625e-001  -1.6582e-002
    5   3.9063e-001   9.7612e-002
    6   3.9844e-001   4.0399e-002
    7   4.0234e-001   1.1876e-002
    8   4.0430e-001  -2.3616e-003
    9   4.0332e-001   4.7552e-003
   10   4.0381e-001   1.1963e-003
   11   4.0405e-001  -5.8279e-004
   12   4.0393e-001   3.0670e-004
   13   4.0399e-001  -1.3805e-004
   14   4.0396e-001   8.4321e-005
   15   4.0398e-001  -2.6867e-005
   16   4.0397e-001   2.8727e-005
   17   4.0397e-001   9.2956e-007
   18   4.0397e-001  -1.2969e-005
   19   4.0397e-001  -6.0197e-006
   20   4.0397e-001  -2.5451e-006
   21   4.0397e-001  -8.0775e-007
   22   4.0397e-001   6.0902e-008
   23   4.0397e-001  -3.7343e-007
   24   4.0397e-001  -1.5626e-007
```

```
  25    4.0397e-001  -4.7680e-008
  26    4.0397e-001   6.6108e-009

ans =

   0.40397275239229

diary off
```


Problem 8: cos(4x*sqrt(1-x^2)) + 1 -8x^2 + 8x^4
2
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

9. $g(x) = 2x - Ax^2$

(a) All you have to do is show the fixed point of $g(x) = \frac{1}{A}$

$$p = g(p) = 2p - Ap^2$$

$$2p - p = Ap^2$$

$$p = Ap^2$$

$$1 = Ap$$

$$\frac{1}{A} = p \implies g\left(\frac{1}{A}\right) = 2\left(\frac{1}{A}\right) - A\left(\frac{1}{A}\right)^2 = \frac{2}{A} - A \cdot \frac{1}{A^2}$$

$$= \frac{2}{A} - \frac{1}{A} = \frac{1}{A}$$

(b) Picard Iteration converges when $|g'(x)| < 1$

$$g'(x) = 2 - 2Ax$$

$$|g'(x)| = |2 - 2Ax| < 1$$

$$-1 < 2(1 - Ax) < 1$$

$$-\frac{1}{2} < 1 - Ax < \frac{1}{2}$$

$$-\frac{3}{2} < -Ax < -\frac{1}{2}$$

$$Ax > \frac{1}{2}$$

$$x > \frac{1}{2} \cdot \frac{1}{A}$$

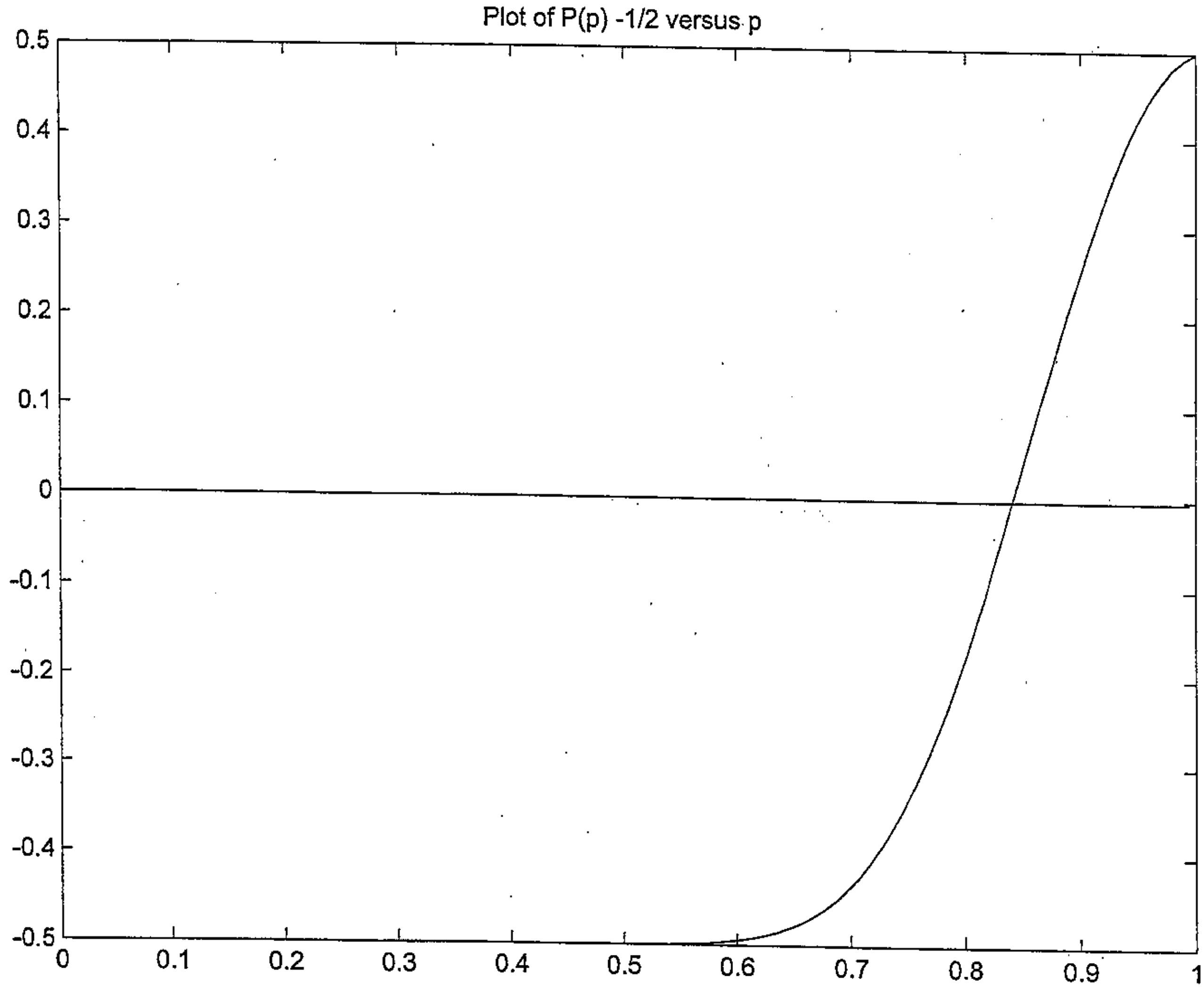
AND

$$Ax < \frac{3}{2}$$

$$x < \frac{3}{2} \cdot \frac{1}{A}$$

$$\frac{1}{2} \cdot \frac{1}{A} < x < \frac{3}{2} \cdot \frac{1}{A}$$

```
type bonus2.m

function [y] = bonus2(p)
y = (.5*(1+p)).*(p./(1-p+p.*p)).^21-.5;
bisect('bonus2',[0.7 .9],1.e-8,1e-8,1)

Bisection iterations for bonus2.m
   it        xm              fm
    1   8.0000e-001   -1.7695e-001
    2   8.5000e-001    3.4398e-002
    3   8.2500e-001   -7.5563e-002
    4   8.3750e-001   -2.1282e-002
    5   8.4375e-001    6.4349e-003
    6   8.4063e-001   -7.4611e-003
    7   8.4219e-001   -5.2163e-004
    8   8.4297e-001    2.9546e-003
    9   8.4258e-001    1.2160e-003
   10   8.4238e-001    3.4704e-004
   11   8.4229e-001   -8.7329e-005
   12   8.4233e-001    1.2985e-004
   13   8.4231e-001    2.1257e-005
   14   8.4230e-001   -3.3036e-005
   15   8.4230e-001   -5.8898e-006
   16   8.4231e-001    7.6835e-006
   17   8.4230e-001    8.9684e-007
   18   8.4230e-001   -2.4965e-006
   19   8.4230e-001   -7.9983e-007
   20   8.4230e-001    4.8504e-008
   21   8.4230e-001   -3.7566e-007
   22   8.4230e-001   -1.6358e-007
   23   8.4230e-001   -5.7538e-008
   24   8.4230e-001   -4.5170e-009
   25   8.4230e-001    2.1994e-008
   26   8.4230e-001    8.7383e-009
   27   8.4230e-001    2.1106e-009

ans =

   0.84230479151011

diary off
```


Plot of P(p) -1/2 versus p
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1