Report on Test 1

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Point Distribution (N=19)

Range	90+	80+	75+	70+	60+	55+	50+	40+	34+	29+	20+	15+	15-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	1	0	1	2	3	2	2	2	5	0	0	1	0

Summary Overall class performance was somewhat disappointing. The mean score was 54. The median score was 56. The high score was 96. The standard deviation was 19. I think this was mainly related to unfamiliarity with the idea that on my exams I am interested in the student demonstrating their understanding of the concepts through explanations and the possibility that the exam itself can be a learning experience and not another chance to just repeat previously completed homework or quiz problems.

#1 Order Notation and Taylor Series. The goal of this problem was **to describe the behavior of the given function** $F(h) = 1 - h + \int_0^h e^{-x^2}dx$ **for very small values of** h**, i.e.** $|h| << 1$**, or as** $h \to 0$**, using the Bachmann-Landau symbol** $\mathcal{O}$. **(a)** The Maclaurin series for $f(x) = e^x = 1 + x + \frac{x^2}{2}$ which means that the Mclaurin series for $e^{-x^2} = 1 - x^2 + \frac{(-x^2)^2}{2}$. **(b)** Thus we can obtain a Mclaurin series for $F(h) = 1 - h + \int_0^h e^{-x^2}dx$ which is $F(h) = 1 - h + \int_0^h (1 - x^2 + \frac{x^4}{4})dx = 1 - \frac{h^3}{3} + h^5 5$ **(c)** Using the result from (b) means that $F(h) = 1 + \mathcal{O}(h^3)$, in other words $p = 3$ and $L = 1$. **(d)** In this question you need to use the original definition of $F(h)$ and the formal limit definition of "big oh" to show that $\lim_{h\to 0} \frac{1 - h + \int_0^h e^{-x^2}dx - L}{h^p}$ equals a constant.

#2 Sequences, Limits, Picard Iteration, Convergence Criteria. The goal of this problem is **to determine the limit of two functional iterations and the relative speed of convergence of two different iteration schemes from given Picard Iteration output, analysis, calculus and/or graphical info.** **(a)** What is the limit of a functional iteration $x_{n+1} = g(x_n)$? The fixed point of $g(x)$! This is because $\lim_{n\to\infty} x_{n+1} = x_\infty$ and $\lim_{n\to\infty} g(x_n) = g(x_\infty)$ So this question is asking to find the fixed points of $A(x) = \sqrt{2+x}$ and $B(x) = \frac{x^2+2}{2x-1}$. **(b)** What determines the rate of convergence of a functional iteration $x_{n+1} = g(x_n)$? The magnitude of $|g'(p)|$, i.e. the size of the derivative AT the fixed point where $p = g(p)$. This derivative can be seen from the given picture and/or explicitly calculated by using Calculus and your answer from part (a). **(c)** What determines whether a functional iteration is linearly convergent or quadratically convergent? Whether $g'(p) = 0$ or not. **(d)** In all of these problems I am interested in the EXPLANATION of your answer, not the answer itself. So, when you say that you know Method 1 is $B(x_n)$ and is faster than Method 2 which is $A(x_n)$, how do you know this? What information are you basing your information on? Note that, this part (d) has 3 parts to it. The stopping criterion must have had a very very small tolerance (clearly smaller than machine epsilon) or perhaps it was just that it asked to stop after a certain number of iterations, say 10. When something asks for 5 decimal places of accuracy, that means the tolerance is `5e-06`. You can then look at the table to determine when each method would have stopped.

#3 These are TRUE or FALSE questions. **(a)** "If $\{p_n\} = 2^{-n}$ is linearly convergent to $p = 0$ and $\{q_n\} = 10^{-(2^n)}$ is quadratically convergent to $q = 0$ then $\lim_{n\to\infty} \frac{q_n - q}{p_n - p} = K, \qquad K > 0$." **FALSE.** Use L'Hopital's Rule to show the given limits of the given sequences produces a value of the limit of zero, which should be unsurprising because it means that q_n is little oh of p_n, in other words q_n converges to its limit faster than p_n does, which is what you would expect a quadratically convergent sequence to do. **FALSE.** The MATLAB command `linspace(a,b,N)` produces a vector that takes the interval between a and b and produces an N component vector for which the first component is a and the N^{th} is b. The MATLAB command $[a : c : b]$ produces a vector where the first component of the vector is a and multiples of increments of size c are added to a until it surpasses or equals b. **(c)** "The **machine precision** i.e., the number ϵ_m such that $1+\epsilon_m = 1$, is the same on all computers." **FALSE.** The machine precision on a computer is directly related to the number of binary digits used to represent the mantissa. Since different computers have different-sized mantissas, different computers have different values of machine precision. Generally on a 32-bit machine the mantissa in a single precision floating point number consists of 24 bits and thus machine precision is approximately 2^{-24} or 10^{-8}.