1. *[30 points total.]* Notation and Taylor Series.

Consider the $F(h) = e^{e^h - 1}$. Given that $F(h) = 1 + \mathcal{O}(h)$, our goal in this problem e limits involving the function $F(h)$ as $h \to 0$.

(a) *[10 p* luate $\lim_{h\to 0^+} \frac{e^{e^h-1}-1}{1}$.

It's continuous at $h=0$ so

$$\lim_{h\to 0^+} = \frac{e^{e^0-1}-1}{1} = \frac{e^{1-1}-1}{1} = \frac{e^0-1}{1} = \frac{1-1}{1} = \boxed{0}$$

$F(h) = 1 + \mathcal{O}(h)$

so $\lim_{h\to 0^+} \frac{(1+\mathcal{O}(h))-1}{1} = \lim_{h\to 0} \mathcal{O}(h) = \boxed{0}$

(b) *[10 points]*. Evaluate $\lim_{h\to 0^+} \frac{e^{e^h-1}-1}{h}$.

L'Hopital's Rule

$$\lim_{h\to 0^+} \frac{e^{e^h-1}\cdot(e^h \cancel{-1})}{1} = \lim_{h\to 0^+} e^{e^0-1}\cdot e^0 = e^0\cdot e^0 = \boxed{1}$$

Use Taylor Series

$$e^{e^h-1}-1 \approx e^{(1+h+\frac{h^2}{2!}+\dots)-1}-1 \approx e^{h+\frac{h^2}{2!}+\dots}-1$$

$$\approx 1+\left(h+\frac{h^2}{2!}+\dots\right)+(h+$$

$$\approx h+\frac{h^2}{2!}+h^2+\dots$$

$$\lim_{h\to 0^+} \frac{h+\frac{h^2}{2!}+\dots}{h} = \boxed{1}$$

(c) *[10 points]*. $\lim_{h\to 0^+} \frac{e^{e^h-1}-1}{h^2}$.

Since e^{e^h-1} is $1+\mathcal{O}(h)$

$$\lim_{h\to 0^+} \frac{e^{e^h-1}-1}{h^2} = \boxed{\infty}$$ since $h^2 \to 0$ faster than $e^{e^h-1}-1 \to 0$

L'Hopital's Rule

$$\lim_{h\to 0^+} \frac{e^{e^h-1}\cdot e^h}{2h} = \frac{e^0\cdot e^0}{0} = \boxed{\infty}$$

[HINT: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$]

2. *[40 points total.]* **Sequences, Limits and Functional Iteration.**

Consider the functional iteration $x_{n+1} = g(x_n) = 2x_n(1-2x_n)$. Our goal in this problem is to completely describe what the limit x_∞ of this iteratively generated sequence will be depending on the starting value x_0.

(a) *[10 points].* For what values of x_0 will $x_0 = x_1 = x_2 = \ldots = x_\infty$? [In other words, find the two fixed points of $g(x)$.]

$$x = 2x(1-2x)$$
$$x = 2x - 4x^2$$
$$0 = x - 4x^2$$
$$0 = x(1-4x)$$
$$x = 0 \quad \text{and} \quad 1-4x = 0$$
$$x = 1/4$$

$(0,0)$ and $(1/4, 1/4)$

$x_0 = 0$ and $x_0 = 1/4$ are fixed points

(b) *[15 points].* **Show experimentally that one of these special values of x_0 in part (a) is an attractive fixed point and one is a repulsive fixed point.** [An attractive fixed point x^* is one in which there exists a $\delta > 0$ so that every sequence starting from an x_0 in the interval $(x^* - \delta, x^* + \delta)$ converges to x^*. A repulsive fixed point x^* is one in which there exists a $\delta > 0$ so that the only sequence starting from an x_0 in an interval $(x^* - \delta, x + \delta)$ that converges to x^* is the one where $x_0 = x^*$.]

Pick values near $x = 0$ and iterate using $x_{n+1} = g(x_n)$

If $x \to 0$ then its attractive, if x never goes to 0 then its repulsive

n	x_n	$g(x_n)$
0	0.01	0.0196
1		0.037663
2		0.069
3		0.1198
4		0.1822
5		0.23
6		0.248
7		0.249993

n	x_n
0	-0.01
1	-0.02
2	-0.042
3	-0.0921
4	-0.2182
5	-0.627
6	-2.25
7	-37

n	x_n
0	0.1
1	0.16
2	0.2176
3	0.245801
4	0.249929
	0.25

Pick values near $x = 1$ and iterate using $x_{n+1} = g(x_n)$

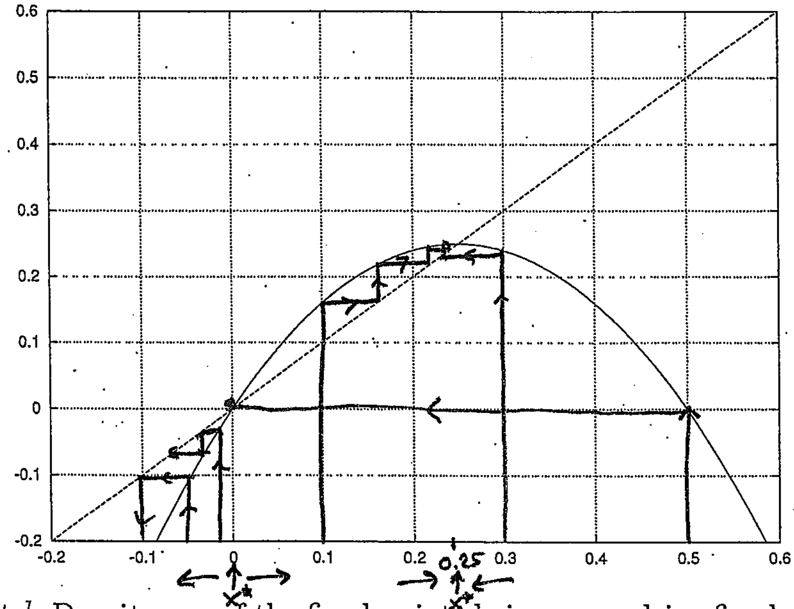
n	x_n
0	0.24
1	0.2496
2	0.249999
3	0.25

n	x_n
0	0.26
1	0.2496
2	0.249995
3	0.25

n	x_n
0	0.2
1	0.24
2	0.2496
3	0.249999
4	0.250

n	x_n
0	0.3
1	0.24
2	0.2496
3	0.249995
4	0.25

Here is a graph of $y = g(x) = 2x(1-2x)$ and $y = x$ on the interval $-0.2 \le x \le 0.6$.



fixed pts at (0,0) repulsive

and (0.25,0.25) attractive

(c) *[5 points].* Despite one of the fixed points being a repulsive fixed point there exists another point on the real number line for which if $x_0 \neq x^*$ the sequence $x_{n+1} = g(x_n) = 2x_n(1-2x_n)$ will still converge to this repulsive point. Find this value x_0.

$x_0 = 0.5$

$x_1 = g(x_0) = g(0.5) = 0$

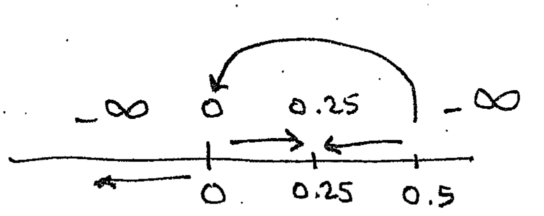
$x_2 = g(x_1) = g(0) = 0$

$x_3 = g(x_2) = g(0) = 0$

$x_\infty = 0$

(d) *[10 points].* Draw a real number line or table below and clearly indicate for which initial values x_0 the sequence will converge to a finite value x_∞ and for which initial values x_0 the sequence will diverge to an infinite value. In other words, make sure you are defining a function which takes any real number x_0 as input and produces x_∞ as output. **Is it possible for the sequence to converge to $+\infty$?**

$$x_\infty = F(x_0) = \begin{cases} -\infty, & x_0 < 0 \\ 0, & x_0 = 0 \\ 0.25, & 0 < x_0 < 0.5 \\ 0, & x_0 = 0.5 \\ -\infty, & x_0 > 0.5 \end{cases}$$



No, the sequence converges to 0, 0.25 or $-\infty$ for any $x_0 \in \mathbb{R}$.

3. *[30 points total.]* **TRUE or FALSE.**

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counter example for which the statement is NOT true is best. If you think the answer is **TRUE** you should also explain why you believe the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

(a) Newton's Method will always converge faster to the solution of $f(x) = 0$ than the Bisection Method will (assuming Bisection is given a bracket containing the root and Newton's initial guess IS "close enough" to the solution).

FALSE

Newton's Method is not globally convergent. If $f'(x) = 0$ near the root Newton's Method will likely diverge or converge very slowly. If x_0 is such that $f'(x_0) = 0$ Newton's will fail immediately

(b) MATLAB can represent (and do calculations with) non-zero real numbers closer to zero than `realmin=2.25e-308`.

TRUE

They are called "denormals" and they are discussed in the textbook.

This is another reason why underflow is not as serious as overflow.

It is NOT true Matlab can represent numbers bigger than realmax.

(c) If two convergent sequences, p_n and q_n are both linearly convergent then they take the same number of steps to be within ϵ of their respective limits p_∞ and q_∞.

FALSE

$p_n = \frac{1}{2^n}$ and $q_n = \frac{1}{n^2}$ are both linearly convergent but p_n approaches its limit faster than q_n (less steps).