Numerical Analysis

Math 370 Fall 2004 — MWF 2:30 - 3:25pm

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Worksheet 18

SUMMARY Successive Over-Relaxation for Linear Systems

Matrix representation of iterative schemes for linear systems

We have written down the matrix implementation of Jacobi and Gauss-Seidel iteration in the form

$$\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$$

and derived how T depends on A and $\vec{c}$ depends on A and $\vec{b}$ for each method.

Gauss-Seidel Iteration

$$\vec{x}_{k+1} = (D - L)^{-1}U\vec{x}_k + (D - L)^{-1}\vec{b}$$

Jacobi Iteration

$$\vec{x}_{k+1} = D^{-1}(L + U)\vec{x}_k + D^{-1}\vec{b}$$

Successive Over-Relaxation (SOR)

$$\vec{x}_{k+1} = (D - \omega L)^{-1}[\omega U + (1 - \omega)D]\vec{x}_k + (D - \omega L)^{-1}\vec{b}$$

Gauss-Seidel ends up being a special case of successive over-relaxation with $\omega = 1$.

Spectral Radius

The spectral radius $\rho(A)$ of a N x N matrix A is defined as $\rho(A) = max|\lambda|$, where λ is an eigenvalue of A.

Properties of the Spectral Radius

(a) $||A||_2 = \sqrt{\rho(A^T A)}$

(b) $\rho(A) \leq ||A||$, for any "natural matrix norm" (i.e. a norm which also applies to vectors)

The importance of the spectral radius of a matrix is that it allows us to say a lot about the convergence and rate of convergence of iterative schemes of the form $\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$

THEOREM

The iterative scheme $\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$ generates a sequence $\{\vec{x}_n\}$ which converges to the unique solution of $\vec{x} = T\vec{x} + \vec{c}$ for any initial guess $\vec{x}_0$ if and only if $\rho(T) < 1$.

COROLLARY

If $||T|| < 1$ for any natural matrix norm and c is a given vector then the iterative scheme $\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$ converges to $\vec{x}$ and the following error bound holds:

$$||\vec{x} - \vec{x}_k|| \leq ||T||^k ||\vec{x}_0 - \vec{x}||$$

A rule of thumb is that

$$||\vec{x} - \vec{x}_k|| \approx \rho(T)^k ||\vec{x}_0 - \vec{x}||$$

Mo' Theorems
We can denote the matrices used by each particular iterative method below:
SOR iteration uses $T_\omega = (D - \omega L)^{-1}[\omega U + (1-\omega)D]$
Jacobi Iteration uses $T_J = D^{-1}(L+U)$
Gauss-Seidel uses $T_G = (D-L)^{-1}U$

Kahan Theorem
If $a_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$ then $\rho(T_\omega) \geq |\omega - 1|$. Therefor SOR will only converge if $0 < \omega < 2$.

Ostrowski-Reich Theorem
If A is a positive definite, tridiagonal matrix then $\rho(T_G) = \rho(T_J)^2 < 1$ and the optimal choice of ω is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_J)]^2}}$$

Positive Definite Matrix
A n by n matrix A is said to be **positive definite** if A is symmetric and if $x^T A x > 0$ for every n-dimensional column vector $x \neq 0$.
Example
Consider the system of equations

$$\begin{array}{ccccccc} 4x & + & 3y & & & = & 24 \\ 3x & + & 4y & - & z & = & 30 \\ & - & y & + & 4z & = & -24 \end{array}$$

Let's try and solve this using Jacobi Iteration, Gauss-Seidel and optimal SOR. Use an initial guess of $(1,1,1)^T$. The exact solution is $(3,4,-5)^T$. Use MATLAB as a tool to assist you. You will want to use `sor.m` in the `linalg` directory of the NMM toolbox.

You will need to find the spectral radius of the system, and determine whether the matrix is positive definite.