Numerical Analysis

Worksheet 13

SUMMARY Solving Systems of Non-Linear Equations, i.e. $\vec{f}(\vec{x}) = 0$
READING Recktenwald, Sec 8.5, pp. 427–445

Introduction
We have spent the last Unit learning techniques of solving the equation $f(x) = 0$ numerically. That is we have been solving non-linear equations in one-variable. Of course, most interesting problems have more than one variable involved. In this next Unit we will learn how to solve systems involving many variables, in the form of non-linear or linear equations.

EXAMPLE

Consider

$$\begin{aligned} y &= \alpha x + \beta \\ y &= x^2 + \sigma x + \tau \end{aligned}$$

This nonlinear system consists of the equations for a line and a parabola, respectively. Our problem is to find the coordinates of the point of intersection for these two curves, for any line and parabola in this form.

1. What are the parameters in this system? What are the variables?

2. Can you write this system in the form $Ax = b$ where A is a 2x2 matrix and x is a 2x1 vector of variables and b is a 2x1 vector?

3. How is this version of $Ax = b$ different from the systems you solved in Math 212/214?

Note in this case we could think of this system as vector root-finding problem, i.e.

$$\vec{f}(\vec{x}) = A\vec{x} - \vec{b} = \vec{0}$$

Similar to the solution technique in solving $f(x) = 0$ we need to find numerical algorithms which generate a sequence of vectors $\{\vec{x}_n\}$ which have as their limit the value of $\vec{x}$ which makes $\vec{f} = 0$, i.e. solves the systems of non-linear equations.

The two most common iterative methods for solving these kinds of systems are called Succesive Substitution, and, Newton's Method (for Systems).

Generic Algorithm for Iterative Solution of Nonlinear Systems

(Input initial guess for solution)
1. LET $x = x^{(0)}$
(Begin Iterating ...)
2. FOR k = 0, 1, 2, ...
(Evaluate the vector function to see how close to the solution we are)
3. $f^{(k)} = f(x^{(k)}) = A(x^{(k)})x^{(k)} - b(x^{(k)})$
(Convergence criterion)
4. IF $||f^{(k)}||$ is ''small enough'', STOP
(Calculate how to modify the current guess: **Will be different for each method**)
5. $\Delta x^{(k)} = ...$
(Produce a new guess from the old guess)
6. $x^{(k+1)} = x^{(k)} + \Delta x^{(k)}$
7. END FOR
8. END PROGRAM

Successive Substitution (Picard Iteration for Vector Functions)
The modify step (LINE 5) in the generic algorithm for iterative solution of nonlinear system becomes
SOLVE $A^{(k)}\Delta x^{(k)} = -f^{(k)}$
Note, that one can combine the modify (LINE 5) and update (LINE 6) steps to produce one step to find your next guess:
SOLVE $A^{(k)}x^{(k+1)} = b^{(k)}$

Newton's Method for Vector Functions
The modify step (LINE 5) in the generic algorithm for iterative solution of nonlinear system becomes
SOLVE $J^{(k)}\Delta x^{(k)} = -f^{(k)}$
where J is the Jacobian of the non-linear system. The Jacobian matrix of a system of non-linear equations is given by

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Exercise
Consider the system

$$\begin{aligned} y &= 1.4x - 0.6 \\ y &= x^2 - 1.6x - 4.6 \end{aligned}$$

This system has two solutions : (-1,-2) and (4,5). Depending on the initial guess, the algorithms will converge to one or the other solution.
EXAMPLE
Find the Jacobian matrix for the given system above.