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# Numerical Analysis

Math 370 Fall 2004  
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MWF 2:30 - 3:25pm  
Fowler North 5

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## Worksheet 8

**SUMMARY** Rates of Convergence of Iterative Sequences

### Linear, Superlinear and Quadratic Convergence of Sequences

#### Definition

Suppose we have a convergent sequence  $\{x_n\}$  which converges to  $x_\infty$ . If there exists a constant  $0 < C < 1$  and an integer  $N$  such that

$$|x_{n+1} - x_\infty| \leq C|x_n - x_\infty|, \text{ for } n \geq N$$

we say  $\{x_n\}$  converges **LINEARLY**.

In general we can say that if the following limit exists with positive constants  $\alpha$  and  $\lambda$ ,

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|^\alpha} = \lambda$$

then, the sequence converges at a **rate of convergence of order**  $\alpha$ , with asymptotic error constant  $\lambda$ . When  $\alpha = 1$  this is called **linear convergence**. When  $\alpha = 2$  this is called **quadratic convergence**. If  $\alpha = 1$  and  $\lambda = 0$  or the following limit exists,

$$\lim_{n \rightarrow \infty} \frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} = 0$$

The sequence is said to converge **superlinearly**.

Let's put all of this together in the following example.

#### Example

Consider  $p_n = n^{-2} = \frac{1}{n^2}$  and  $q_n = \frac{1}{2^n} = 2^{-n}$ .

1. What is the limit of each of the sequences?
2. For each of the sequences, find out how many steps it takes to be within  $10^{-4}$  of its limit.
3. In terms of "big oh" and "little oh" notation, can you write down a relationship between  $q_n$  and  $p_n$ ?

4. Does  $p_n$  converge linearly? superlinearly? quadratically?

5. Does  $q_n$  converge linearly? superlinearly? quadratically?

6. Which sequence converges faster to its limit? Explain your answer.

GROUPWORK
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Example 1 Show that  $r_n = \frac{1}{n^n}$  converges superlinearly to zero.

Example 2 Show that  $s_n = \frac{1}{10^{2^n}}$  converges quadratically to zero.

**NOTE: Algorithms which produce sequence of approximation which converge quadratically are extremely rare.**