2000

1. *[20 points total.]* **Interpolation.**
The following data is used in Question 1 and 2.

i	x_i	y_i
0	-1	1/2
1	0	1
2	1	2

Besides Lagrange Interpolating Polynomials there are many other types of basis functions which can be used to form interpolating polynomials. Consider the Newton Interpolating Polynomial, $N(x)$. It has the form

$$N(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \dots$$

(a) *[3 pts]* For the given data, write down the general Newton interpolating Polynomial of degree 2.

$$N(x) = a_0 + a_1(x - {}^-1) + a_2(x - {}^-1)(x - 0)$$
$$= a_0 + a_1(x+1) + a_2(x+1)x$$

(b) *[15 pts]* For the given data, find the second degree Newton interpolating Polynomial $N(x)$.

$a_0 = 2$
$a_1 + \cancel{a_0} = 1$
$a_1 = -1$
$a_2 =$

$$N(x_0) = y_0 \Rightarrow a_0 = \tfrac{1}{2}$$
$$N(x_1) = y_1 \Rightarrow 1 = a_0 + a_1(1)$$
$$N(x_2) = y_2 \Rightarrow 2 = a_0 + a_1(2) + a_2(1)(2)$$

$$a_1 = 1 - \frac{1}{2} = \frac{1}{2}$$
$$2a_2 = 2 - \frac{1}{2} - 2\left(\frac{1}{2}\right) = \frac{1}{2}$$
$$a_2 = \frac{1}{4}$$

(c) *[2 pts]* Recall, the form of the Lagrange Interpolation Polynomial (from Quiz 9) was $L(x) = y_0 L_{2,0}(x) + y_1 L_{2,1}(x) + y_2 L_{2,2}(x) = 1 + \frac{3}{4}x + \frac{1}{4}x^2$. How is $L(x)$ related to $N(x)$?

They must be the same

$$N(x) = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}x(x+1)$$
$$L(x) = 1 + \frac{3}{4}x + \frac{1}{4}x^2$$

2000

2. *[40 pts. total]* **Approximation Theory.**

This problem involves choosing the curve of best fit for the given data from **(1)**. The choices are between $y = ax + b$ and $y = \dfrac{1}{cx+d}$.

(a) *[10 pts]* Compute the value of a which minimizes the least square error between $y = ax + b$ and the given data.

x	y	xy	x^2
-1	$1/2$	$-1/2$	$1/4$
0	1	0	0
1	2	2	1

$\bar{x} = 0 \quad \bar{y} = \frac{7}{6} \quad \overline{xy} = \frac{1}{2}$

$\overline{x^2} = \cancel{\frac{5}{12}}\ 2/3$

$a = \dfrac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2} = \dfrac{1/2 - 0}{2/3 - 0} = \cancel{\frac{9}{10}}\ \frac{3}{4}$

(HINT: Do not use any decimal approximations of fractions or logarithms.)

(b) *[10 pts]* Compute the value of b which minimizes the least square error between $y = ax + b$ and the given data.

$b = \bar{y} - a\bar{x}$

$= \bar{y} = \frac{7}{6}$

$y = \frac{3}{4}x + \frac{7}{6}$

$y = \frac{9x + 14}{12}$

$P(0) = \cancel{\frac{35}{30}}\ \frac{14}{12}$

$P(1) = \frac{23}{12}$

$P(-1) = \frac{5}{12}$

Least Square Error

$\left(\frac{14}{12} - 1\right)^2 + \left(\frac{23}{12} - 2\right)^2 + \left(\frac{5}{12} - \frac{1}{2}\right)^2$

$\left(\frac{2}{12}\right)^2 + \left(\frac{1}{12}\right)^2 + \left(\frac{1}{12}\right)^2 = \frac{6}{144}$

2999

(c) *[4 pts]* Convert the form of the equation from a non-linear form to a linear form; that is from $y = \dfrac{1}{cx+d}$ to $Y = AX + B$. Write down relationships between y, x, c, d and Y, X, A, B, respectively.

$y = \dfrac{1}{cx+D}$ $\quad \dfrac{1}{y} = cx + d$

$Y = \dfrac{1}{y}$

$A = c$

$B = d$

$X = x$

(d) *[10 pts]* Compute the value of A which minimizes the least square error between $Y = AX + B$ and the given data.

	x	y	Y	xY	x^2
	-1	$1/2$	2	-2	1
	0	1	1	0	0
	1	2	$1/2$	$1/2$	1
means	0	$7/6$	$7/6$	$-1/2$	$2/3$

$$A = \frac{\overline{XY} - \bar{X}\bar{Y}}{\overline{X^2} - \bar{X}^2} = \frac{-1/2 - 0}{2/3 - 0} = -3/4$$

(b) *[10 pts]* Compute the value of B which minimizes the least square error between $Y = AX + B$ and the given data.

$$B = \bar{Y} - A\bar{X} = \bar{Y} = 7/6$$

$$Y = -\frac{3}{4}X + \frac{7}{6}$$

$$y = \frac{1}{-\frac{3}{4}x + \frac{7}{6}} = \boxed{\frac{12}{-9x+14} = y}$$

$Q(0) = \dfrac{12}{14}$

$Q(1) = \dfrac{12}{5}$

$Q(-1) = \dfrac{12}{23}$

error $= \left(\frac{12}{14} - 1\right)^2 + \left(\frac{12}{5} - 2\right)^2 + \left(\frac{12}{23} - \frac{1}{2}\right)^2$

(e) *[6 pts]* Which curve of best fit would you choose to approximate the data? Why?

The curve which has the smallest least squares error.

$\left(\frac{1}{46}\right)^2 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{14}\right)^2$ is error from the curve

$\left(\frac{2}{12}\right)^2 + \left(\frac{1}{12}\right)^2 + \left(\frac{1}{12}\right)^2$ is error from the line. The LINE does better than the curve

2DDP

3. *[30 pts. total]* **Systems of Equations.**
Use Newton's Method with an initial guess of $\vec{x}_0 = (0,0,0)^T$ and find two approximations, $\vec{x}_1$ and $\vec{x}_2$ to the exact solution to $\vec{f}(\vec{x}) = 0$, where

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix} = \begin{pmatrix} -2x + xyz + 1 \\ 2y + xy + yz - 4 \\ -4z + y^2 + 3 \end{pmatrix} .= \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Use Newton's Method with an initial guess of $\vec{x}_0 = (0,0,0)^T$ and find an approximation, $\vec{x}_1$ to the exact solution to $\vec{f}(\vec{x}) = 0$.

(a) *[10 pts]* Compute the jacobian matrix.

$$J = \begin{pmatrix} -2 + yz & xz & xy \\ y & 2+x+z & y \\ 0 & 2y & -4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$

2000

(b) *[10 pts]* Use Newton's Method to find your first approximation, $\vec{x}_1$

$$J(\vec{x}_0)(\vec{x}_1 - \vec{x}_0) = -f(\vec{x}_0)$$

$$J(\vec{0}) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix} \qquad \vec{x}_0 = \vec{0}$$

$$+\vec{f}(\vec{0}) = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \Rightarrow -\vec{f}(\vec{0}) = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix} \vec{x}_1 = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{pmatrix} 1/2 \\ 2 \\ 3/4 \end{pmatrix}$$

(d) *[5 pts]* Show that the vector $\vec{r} = (1,1,1)^T$ is the root of $\vec{f}(\vec{x})$

$$\vec{f}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+1+1 \\ 2+1+1-4 \\ -4+1+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \checkmark$$

(e) *[5 pts]* Evaluate $||\vec{x}_1 - \vec{r}||_1$, $||\vec{x}_1 - \vec{r}||_2$ and $||\vec{x}_1 - \vec{r}||_\infty$. Do you consider your solution "converged"? What tolerance would you have to use to consider $\vec{x}_1$ a reasonable approximation to $\vec{r}$?

$$\left\|\begin{pmatrix} 1/2 \\ 2 \\ 3/4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\|_1 = \left\|\begin{matrix} -\frac{1}{2} \\ 1 \\ \cancel{2}/4 \end{matrix}\right\|_1 = \frac{\cancel{2}}{4} + 1 + \frac{1}{2} = 7/4$$

~~Tolerance~~

The sol^n is NOT converged!

$$\left\|\begin{matrix} -\frac{1}{2} \\ 1 \\ \cancel{1}/4 \end{matrix}\right\|_\infty = 1$$

$$\left\|\begin{matrix} -\frac{1}{2} \\ 1 \\ \cancel{3}/4 \end{matrix}\right\|_2 = \sqrt{\frac{1}{4} + 1 + \frac{\cancel{9}}{16}} = \frac{\cancel{29}}{\cancel{4}} = \cancel{\frac{\sqrt{29}}{25}}$$

$$\sqrt{\frac{4}{16} + \frac{16}{16} + \frac{1}{16}} \qquad \frac{1}{4}\sqrt{21}$$