Differential Equations

Math 341 Fall 2014 MWF 3:00-3:55pm Fowler 307
 http://faculty.oxy.edu/ron/math/341/14/

Worksheet 28

TITLE Laplace Transforms and Introduction to Convolution
CURRENT READING Blanchard, 6.5

Homework #11 Assignments due Monday November 17
Section 6.1: 2, 3, 5, 7, 8, 9, 15, 18, 25*.
Section 6.2: 1, 2, 4, 8, 15, 16, 18*.
Homework #12 Assignments due Monday November 24
Section 6.3: 5, 6, 8, 15, 18, 27, 28.
Section 6.4: 1, 2, 6, 7*.

SUMMARY
We shall discuss the equivalent of the product rule for Laplace Transforms and be introduced to the concept of the convolution of two functions.

1. Product Rule for Laplace Transforms

DEFINITION: **convolution**

If two functions $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ then **the convolution of** f **and** g, usually denoted $f * g$ is defined to be

$$\int_0^t f(\tau)g(t-\tau)d\tau$$

NOTE: this "product" is a function of t. The use of the "*" symbol is deliberate, since the convolution operation has the following familiar properties:

THEOREM: properties of convolution

If f, g and h are piecewise continuous on $[0, \infty)$, then
I. $f * g = g * f$ (Commutative)
II. $f * (g + h) = f * g + f * h$ (Distributive Under Addition)
III. $f * (g * h) = (f * g) * h$ (Associative)
IV. $f * 0 = 0$ (Existence of Zero Object)

THEOREM: **The convolution theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order so that $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$ then $\mathcal{L}[f * g] = F(s)G(s)$.

Corollary

$\mathcal{L}^{-1}[F(s)G(s)] = f * g$.

Exercise Evaluate $\mathcal{L}\left[\int_0^t e^\tau \sin(t-\tau)d\tau\right]$. [HINT: use the convolution theorem!]

2. The General Solution To A Non-Homogeneous Linear Second Order ODE

Consider $y'' + py' + qy = f(t)$ with $y(0) = 0$ with $y'(0) = 0$. It can be shown that the exact solution $y(t)$ (for $t > 0$) to this problem is given by a convolution, namely $\xi(t) * f(t)$ where $\mathcal{L}[\xi(t)] = \dfrac{1}{s^2 + ps + q}$

This is a pretty incredible result. It is known as **Duhamel's Principle**.

What one does is show that the $y = \xi(t)$ is the solution to the **unit impulse** version of the problem, i.e. $y'' + py' + qy = \delta_0(t)$ with $y(0) = 0$ with $y'(0) = 0$. Thus, if one wants to solve any other non-homogeneous problem all one needs to do is solve the unit impulse problem, and **convolve** that solution with the given non-homogeneous function.

EXAMPLE

Show that the general solution to $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$ is
$y(t) = \displaystyle\int_0^t \sin(t - u)\ f(u)\ du$

3. Derivatives of Laplace Transforms

EXAMPLE

Show that $\frac{d}{ds}F(s) = -\mathcal{L}[tf(t)]$ and $\frac{d^2}{ds^2}F(s) = \mathcal{L}[t^2 f(t)]$.

THEOREM: Derivatives of Laplace Transforms

When $F(s) = \mathcal{L}[f(t)]$, and $n = 0, 1, 2, \ldots$ $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$

Exercise We now have TWO different ways to show that $\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}$ WITHOUT EVALUATING AN INTEGRAL. Do this.

4. Laplace Transform of an Integral

We can use the Convolution Theorem with $g(t) = 1$ and show that $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \dfrac{F(s)}{s}$

NOTE

I. Multiplication of $f(t)$ by t generally involves **differentiation** of its Laplace Transform $F(s)$ with respect to s.

II. Division of $F(s)$ by s generally involves **anti-differentiation** of its Inverse Laplace Transform $f(t)$ with respect to t

Mathematically, these statements can be expressed as

I. $\mathcal{L}[tf(t)] = \dfrac{d}{ds}\left(\mathcal{L}[f(t)]\right)$ and **II.** $\mathcal{L}[f(t)] = \dfrac{\mathcal{L}[f'(t)]}{s}$ and are always true if the Function-Transform pair $f(t) \leftrightarrow F(s)$ has the property that $f(0) = 0$.

GroupWork

Find $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+1)}\right]$