Differential Equations

Math 341 Fall 2014 — MWF 3:00-3:55pm Fowler 307

©2014 Ron Buckmire — http://faculty.oxy.edu/ron/math/341/14/

Worksheet 27

TITLE The Laplace Transform and Second Order ODEs
CURRENT READING Blanchard, 6.3-6.4

Homework #11 Assignments due Monday November 17
Section 6.1: 2, 3, 5, 7, 8, 9, 15, 18, 25*.
Section 6.2: 1, 2, 4, 8, 15, 16, 18*.
Homework #12 Assignments due Monday November 24
Section 6.3: 5, 6, 8, 15, 18, 27, 28.
Section 6.4: 1, 2, 6, 7*.

SUMMARY
We shall learn how to apply Laplace Transforms to solve second-order ordinary differential equations of the form $y'' + py' + qy = f(t)$ and especially ones that have a brand-new wild and wacky mathematical object called the Dirac Delta Function. We shall also become more familiar with `Mathematica`.

RECALL **Zill, Example 3, page 295.**
Let's use `Mathematica` to show that the solution of $y''-6y'+9y = t^2e^{3t}$, $y(0) = 2$, $y'(0) = 17$ is $y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t}$.

1. Using Mathematica to Solve ODEs

First of all, `Mathematica` can be used to solve this ODE directly.

The command to use is `DSolve`. Specifically

```
DSolve[ {y''[t]-6y[t]+9y[t]==t^2 Exp[ 3 t ],y[0]==2, y'[0]==17 },y[t],t]
```

Type the above command verbatim and press `SHIFT-ENTER` to look at the results!

We can also use `Mathematica` to find Laplace Transforms that we need to solve the problem.

```
LaplaceTransform[{y''[t]-6y[t]+9y[t]==t^2 Exp[ 3 t ]},t,s]

Solve[%, LaplaceTransform[y[t], t, s]]

InverseLaplaceTransform[%, s, t]
```

So, the commands to remember are `DSolve`, `InverseLaplaceTransform[F[s], s, t]` and `LaplaceTransform[y[t], t, s]`.
NOTE the different order of s and t in the two commands!

Also, the Heaviside Function in Mathematica is called `HeavisideTheta[x]`.

2. Applications of Laplace Transforms to Linear Second-Order ODEs

GroupWork

Solve the following initial value problems using Laplace Transforms (and `Mathematica`)!

Zill, page 303, #31.

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1 \text{ where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & 2\pi \le t \end{cases}$$

Blanchard, page 600, #29. $y'' - 4y' + 5y = 2e^t, \quad y'(0) = 1, \quad y(0) = 3.$

3. The Unit Impulse Function

Consider the unit impulse function $\delta_a(t) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a < t < t_0 + a \\ 0, & t_0 + a < t \end{cases}$

DEFINITION: **Dirac Delta Function** The **Dirac Delta Function** is denoted by $\delta(t-t_0)$ and is the object (it's not really a function) which results when one takes the limit as $a \to 0$ of the unit impulse function $\delta_a(t-t_0)$. In other words, $\delta(t-t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$.

The Dirac Delta Function also has the property that $\displaystyle\int_{-\infty}^{\infty} \delta(t-t_0)\, dt = 1$

THEOREM: **The Laplace Transform of the Dirac Delta Function**

For $t_0 > 0$, $\mathcal{L}[\delta(t-t_0)] = e^{-st_0}$ and $\mathcal{L}^{-1}[e^{-st_0}] = \delta(t-t_0)$. (For more details, see Blanchard, p. 597).

Interestingly, we can relate the Heaviside function $\mathcal{H}(t)$ and Dirac Delta Function $\delta(t)$. Consider the following integrally defined function $f(x) = \displaystyle\int_{-\infty}^{x} \delta(t-t_0)\, dt$.

Q: What does $f(x)$ look like?
A: Depends on the relationship between x and t_0. How? Can you draw a picture of it?

The integral of the ____________________ is the ____________________, and the ____________________ of Heaviside Function is equal to the Dirac Delta Function. (Pretty cool, eh?) In the space below, sketch the Heaviside Function $\mathcal{H}(t)$ and Dirac Delta Function $\delta(t)$ for all t values.

The Delta Function in Mathematica is called `DiracDelta[x]`.

4. Delta Function as Source Term

What's interesting about the Dirac Delta Function is that it allows us to model situations where an instantaneous impulse is applied to a system at a certain time. Laplace Transforms are really the only technique which allow solution of such initial value problems.

EXAMPLE

Zill, page 316, Example 1. Solve $y'' + y = 4\delta(t - 2\pi)$ where
(a) $y(0) = 1, \quad y'(0) = 0$ and **(b)** $y(0) = 0, \quad y'(0) = 0$ [HINT: Do (b) first!]

Exercise

In the space below, sketch the solutions to the initial value problems from the previous example, i.e. $y'' + y = 4\delta(t - 2\pi), \quad y(0) = 1, y'(0) = 0$ and
$y'' + y = 4\delta(t - 2\pi), \quad y(0) = 0, y'(0) = 0$