Differential Equations

Worksheet 24

TITLE Dissipative Systems
CURRENT READING Blanchard, 5.3 & 5.4

Homework Assignments due Monday November 10
(* indicates EXTRA CREDIT)
Section 5.1: 3, 4, 5, 8, 18, 20*.
Section 5.3: 2, 9, 12, 13, 14, 18*.
Chapter 5 Review: 1, 2, 6, 7, 8, 9, 11, 12, 26, 27*.

SUMMARY
We shall continue our analysis of non-linear systems by introducing the concept of a Lyapunov function and learn about gradient systems.

EXAMPLE

Recall the ODE for the damped harmonic oscillator $y'' + py' + qy = 0$ written as a system of ODEs

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -pv - qy \end{aligned}$$

Recall that the function $H(y, v) = \frac{1}{2}v^2 + \frac{1}{2}qy^2$ is a Hamiltonian for the system when $p = 0$. However, what is $\frac{dH}{dt}$ now?

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt} \\ &= (qy)v + v(-pv - qy) \\ &= -pv^2 \end{aligned}$$

Which, when $p > 0$ implies that the quantity $H(y, v) = \frac{1}{2}v^2 + \frac{1}{2}qy^2$ decreases with time along solution curves of the given system. Such a function is not known as a Hamiltonian function but a Lyapunov function. Lyapunov functions are often used to make conclusions about the stability of equilibria of nonlinear systems of DEs.

DEFINITION: Lyapunov Function

A function $L(x, y)$ is called a **Lyapunov function** for a system of differential equations, if, for every solution $(x(t), y(t))$ that is not an equilibrium solution of the system,

$$\frac{d}{dt}L(x(t), y(t)) \leq 0$$

for every t with strict inequality except for a discrete set of values for t.

1. Gradient Systems

A system of differential equations is known as a **gradient system** if there exists a function $G(x, y)$ such that for every (x, y)

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial G}{\partial x} \\ \frac{dy}{dt} &= \frac{\partial G}{\partial y} \end{aligned}$$

If $(x(t), y(t))$ are solutions of gradient system, then

$$\begin{aligned} \frac{dG}{dt} &= \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial G}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial G}{\partial y} \\ &= (G_x)^2 + (G_y)^2 \\ &\geq 0 \end{aligned}$$

This should makes us realize that if we want to form a Lyapunov function for a gradient system all we need to do is select $L(x, y) = -G(x, y)$! So, **all gradient systems possess a Lyapunov function**. (The converse is NOT true, i.e. every system with a Lyapunov function is NOT a gradient system.)

EXAMPLE

Let's show that $L(x, y) = -G(x, y)$ is a Lyapunov function for any gradient system $\dot{x} = G_x, \quad \dot{y} = G_y$.

NOTE

To check whether the system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is a gradient system just check whether $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. To check whether system is Hamiltonian, you check whether $\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$.

Exercise

Blanchard, Devaney & Hall, 5.4.1, page 524.

Consider

$$\frac{dx}{dt} = -x^3$$
$$\frac{dy}{dt} = -y^3$$

(a) Show that $L(x, y) = \frac{1}{2}(x^2 + y^2)$ is a Lyapunov function for the given system. [Is this system a gradient system?]

(b) Sketch the level sets of $L(x, y)$

(c) What can you conclude about the phase portrait of the system given your information from (a) and (b)? [Think about what happens to solutions as t goes to infinity?]

2. Properties of Gradient Systems

Gradient Systems can not possess periodic solutions!

This is a very important result because often when one is analyzing a system quantitatively one wants to determine whether periodic solutions are possible or not. With gradient systems, one knows that it is **not possible** to have a periodic solution (i.e. closed orbit in phase portrait).

By using Linearization, we can show that the eigenvalues of the Jacobian of a gradient system evaluated at its equilibria will always be real (i.e. not complex) and thus solution curves of gradient systems will never be periodic.

Not All Systems That Have Lyapunov Functions Are Gradient Systems

EXAMPLE

The following system has a Lyapunov function of $L(x, y) = x^2 + y^2$ but is NOT a gradient system.

$$\frac{dx}{dt} = -x + y$$
$$\frac{dy}{dt} = -x - y$$

Let's Prove This Result.