Differential Equations

Math 341 Fall 2014 MWF 3:00-3:55pm Fowler 307
 http://faculty.oxy.edu/ron/math/341/14/

Worksheet 23

TITLE Hamiltonian Systems
CURRENT READING Blanchard, 5.1 & 5.3

Homework Assignments #10 due Monday November 10
(* indicates EXTRA CREDIT)
Section 5.1: 3, 4, 5, 8, 18, 20*.
Section 5.3: 2, 9, 12, 13, 14, 18*.
Chapter 5 Review: 1, 2, 6, 7, 8, 9, 11, 12, 26, 27*.

SUMMARY
We shall continue our analysis of non-linear systems by introducing the concept of a Hamiltonian function.

Consider the following nonlinear planar system of ODEs

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^2\end{aligned}$$

Exercise

Show that the function $H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$ has the property that $\frac{dH}{dt} = 0$ if x and y simultaneously satisfy the given system of ODEs. (HINT: Use the Differentiation Chain Rule!)

1. The Hamiltonian

DEFINITION: Hamiltonian function

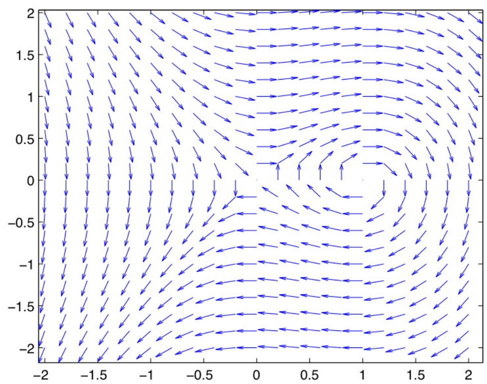
A real-valued function $H(x,y)$ is considered to be a conserved quantity for a system of ordinary differential equations if it is constant along ALL solution curves of the system. In other words, IF $(x(t), y(t))$ is a solution of the system then $H(x(t), y(t)$ is constant for all time which also implies that $\frac{d}{dt}H(x(t), y(t)) = 0$. The function $H(x,y)$ is known as the Hamiltonian function (or Hamiltonian) of the system of ODEs.

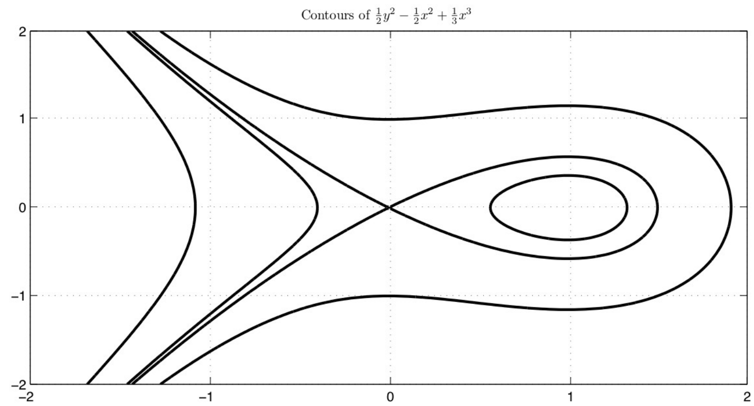
2. The Hamiltonian Level Curves and The Phase Portrait

RECALL

The **level curves** or **contours** of the function $H(x,y)$ are the set of points in the plane which satisfy the equation $H(x,y) = k$ for certain real values k.

Let's compare the level curves of $H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$ with the direction field of the system $\dot{x} = y; \quad \dot{y} = x - x^2$. What do you notice?





3. Hamiltonian System

DEFINITION: Hamiltonian System

A system of differential equations is called a **Hamiltonian system** if there exists a real-valued function $H(x,y)$ such that

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} \end{aligned}$$

for all x and y. The function H is called the Hamiltonian function for the system.

EXAMPLE

The Hamiltonian often has a physical meaning for the sysem of ODEs that is modelling a partcular real-world situation, since it represents a quantity that is being conserved over time. They are sometimes also called **conservative systems**. For example, consider the system of ODEs that represents the **undamped** harmonic oscillator $y'' + qy = 0$:

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -qy \end{aligned}$$

Let's show that the Hamiltonian for this system is $H(y,v) = \frac{1}{2}v^2 + \frac{q}{2}y^2$ which represents the total energy of the oscillator.

4. Obtaining Hamiltonians For Systems

In general the planar nonlinear system of first order DEs looks like

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

In order to find $H(x,y)$ we need to solve the following equations

$$\begin{aligned} f(x,y) &= \frac{\partial H}{\partial y} \\ g(x,y) &= -\frac{\partial H}{\partial x} \end{aligned}$$

Does a Hamiltonian exist for this system? Well, if it does (and H has continuous second partial derivatives) then $\dfrac{\partial^2 H}{\partial x \partial y} = \dfrac{\partial^2 H}{\partial y \partial x}$ which would mean that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} H_y = \frac{\partial}{\partial y} H_x = -\frac{\partial g}{\partial y}$$

So in order to check whether a given system of ODEs has a Hamiltonian or not all one needs to do is check whether

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$

Exercise

Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\begin{aligned} \frac{dx}{dt} &= x + y^2 \\ \frac{dy}{dt} &= y^2 - x \end{aligned}$$

EXAMPLE

Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\begin{aligned} \frac{dx}{dt} &= -x\sin(y) + 2y \\ \frac{dy}{dt} &= -\cos(y) \end{aligned}$$

5. Equilibria of Hamiltonian Systems

Hamiltonian Systems Can Never Have Sources or Sinks As Equilibria.

This is a very significant result because it means that conservative systems do not have "attractive or repulsive fixed points." This allows one to analyze and predict the long-term behavior of such systems analyytically.

Consider

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} \end{aligned}$$

at the point (x_0, y_0) which is the equilibrium point. Let's use the Linearization Technique to prove this important result.

The Jacobian of the linearized version of the Hamiltonian System at (x_0, y_0) will be

What is the trace and determinant of the Jacobian matrix evaluated at (x_0, y_0)?

What are the eigenvalues of $J(x_0, y_0)$?

What do the eigenvalues allow us to conclude about behavior of the system with Hamiltonians near these fixed points?