# Differential Equations

Math 341 Fall 2014 MWF 3:00-3:55pm Fowler 307

 http://faculty.oxy.edu/ron/math/341/14/

## *Worksheet 20*

**TITLE** Linear Systems with Repeated Eigenvalues
**CURRENT READING** Blanchard, 3.5

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**Homework #9 Assignments due Monday November 3**
(* indicates EXTRA CREDIT)
**Section 3.5**: 3, 4, 9, 10, 12, 18*, 23*.
**Section 3.7**: 1,2*,6.
**Chapter 3 Review**: 3, 4, 6, 10, 13, 20*.

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**SUMMARY**
We'll continue to explore the various scenarios that occur with linear systems of ODEs. This time dealing with those that possess repeated eigenvalues. This will involve the introduction of a new concepts, the Generalized Eigenvector. We will also review some important concepts from Linear Algebra, such as the Cayley-Hamilton Theorem.

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## 1. Repeated Eigenvalues

Given a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the characteristic polynomial is $p(\lambda) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \text{tr(A)}\lambda + \det(\text{A}) = 0$.

Previously we showed that the condition for repeated eigenvalues was $(a-d)^2 = -4bc$. In this case there will be only one solution to the quadratic equation, i.e. repeated eigenvalues equal to $\lambda = \dfrac{(a+d)}{2}$.

When there are two eiegenvalues and eigenvectors the general solution to $\dfrac{d\vec{x}}{dt} = A\vec{x}$ is $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$ where $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$, i.e $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to eigenvalues $\lambda_1$ and $\lambda_2$.

**The Easy Case**
**Q:** What do we do if our one eigenvalue has two eigenvectors? (Is this even possible?)
**A:** As long as we have two eiegenvectors we can use the above formula for the general solution. In this case the problem is even simpler because if the eigenspace is 2-dimensional then every vector in $\mathbb{R}^2$ is an eigenvector so the easiest choice is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This situation is possible if the matrix has the form $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$.

**The Hard Case**
**Q:** So what do we do if we only have one eigenvalue $\lambda$ (and only one eigenvector $\vec{v}$), i.e. $\vec{x}_1(t) = e^{\lambda t}\vec{v}$?
**A:** We need to find another vector function $\vec{x}_2(t)$ that is linearly independent to $\vec{x}_1(t)$ at every point $t$.
The answer turns out to be $\vec{x}_2(t) = e^{\lambda t}(\vec{w} + t\vec{v})$ where $(A - \lambda\mathcal{I})\vec{w} = \vec{v}$. In this formula $\vec{v}$ is an eigenvector of $A$ and $\vec{w}$ is a **generalized eigenvector** of rank 2.

DEFINITION: **generalized eigenvector**

An eigenvector $\vec{w}$ associated with $\lambda$ such that $(A - \lambda\mathcal{I})^r\vec{w} = \vec{0}$ but $(A - \lambda\mathcal{I})^{r-1}\vec{w} \neq \vec{0}$ is called a **generalized eigenvector of rank r**.

PROOF

Let's confirm that $\vec{x}(t) = e^{\lambda t}(\vec{w} + t\vec{v})$ is another solution to the ODE.

$$\begin{aligned}
\frac{d\vec{x}}{dt} &= A\vec{x} \\
\frac{d[e^{\lambda t}(\vec{w} + t\vec{v})]}{dt} &= A[e^{\lambda t}(\vec{w} + t\vec{v})] \\
\lambda e^{\lambda t}(\vec{w} + t\vec{v})] + e^{\lambda t}\vec{v} &= e^{\lambda t}[A\vec{w} + A\vec{v}t] \\
e^{\lambda t}(\lambda\vec{w} + \vec{v}) + te^{\lambda t}\lambda\vec{v} &= e^{\lambda t}(A\vec{w}) + (A\vec{v})te^{\lambda t}
\end{aligned}$$

Equating the $e^{\lambda t}$ terms produces the equation $\lambda\vec{w}+\vec{v} = A\vec{w}$, i.e. $\vec{v} = A\vec{w}-\lambda\vec{w} = (A-\lambda\mathcal{I})\vec{w}$
Equating the $te^{\lambda t}$ terms produces the equation $\lambda\vec{v} = A\vec{v}$

So, if we choose $\vec{v}$ and $\vec{w}$ to have these properties then $\vec{x}(t) = e^{\lambda t}(\vec{w} + t\vec{v})$ will solve $\frac{d\vec{x}}{dt} = A\vec{x}$. Yay! The general solution will be $\vec{x} = c_1e^{\lambda t}\vec{v} + c_2e^{\lambda t}(\vec{w} + t\vec{v})$.

**RECALL**

The Cayley-Hamilton Theorem states that a $n \times n$ matrix $A$ satisfies its own characteristic polynomial. In other words, given $p(\lambda) = \det(A - \lambda\mathcal{I}) = 0$, $p(A) = \mathcal{O}$ where $\mathcal{I}$ is the $n \times n$ identity matrix and $\mathcal{O}$ is the $n \times n$ zero matrix. (What an awesome result!)

Since we know there is only one (repeated) eigenvalue $\lambda$, we know that the characteristic polynomial has the form $p(x) = (x - \lambda)^2 = 0$ which means that $p(A) = (A - \lambda\mathcal{I})^2 = \mathcal{O}$.

$$\begin{aligned}
(A - \lambda\mathcal{I})^2 &= \mathcal{O} && \text{(From the Cayley-Hamilton Theorem)} \\
(A - \lambda\mathcal{I})^2\vec{w} &= \mathcal{O}\vec{w} && \text{(Multiply both sides by an unknown vector } \vec{w}\text{)} \\
(A - \lambda\mathcal{I})[(A - \lambda\mathcal{I})\vec{w}] &= \vec{0} && \text{(Group terms and name the bracketed term } \vec{v}\text{)} \\
(A - \lambda\mathcal{I})\vec{v} &= \vec{0} && \text{(Either } \vec{v} = \vec{0} \text{ or it's a generalized eigenvector of } A\text{)}
\end{aligned}$$

**RECALL**

The definition of an eigenvector is a vector $\vec{x}$ which lies in the nullspace of $A - \lambda\mathcal{I}$ (also known as the eigenspace $E_\lambda$), i.e. it solves the equation $(A - \lambda\mathcal{I})\vec{x} = \vec{0}$.

So from the Cayley-Hamilton Theorem we know that the vector $(A-\lambda\mathcal{I})\vec{w}$ lies in the one-dimensional eigenspace $E_\lambda$, i.e. it must be a scalar multiple of the non-zero eigenvector $\vec{v}$.
We still do not know the exact value of vector $\vec{w}$ but we can use the above information to compute it by solving the linear system $(A - \lambda\mathcal{I})\vec{w} = \vec{v}$.

**Algebraic Multiplicity Is Less Than Or Equal To Geometric Multiplicty**
**Algebraic multiplicity** of an eigenvalue is the number of times an eigenvalue satisfies the characteristic polynomial.
**Geometric multiplicity** of an eigenvector is the dimension of the corresponding eigenspace (or the number of eigenvectors corresponding to a particular eigenvalue).

**Exercise**

Given $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ find the eigenvalue(s) and eigenvector(s) of $A$ and confirm that this matrix satisfies the Cayley-Hamilton Theorem.

EXAMPLE

We'll show that $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \vec{x}$ has the general solution $\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \left( \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

**Exercise**

Consider the BONUS Question from **Exam #1, Fall 2009**. Find the general solution of $\frac{dx}{dt} = x + 3y, \quad \frac{dy}{dt} = y$. It turns out that the solution is $x(t) = c_1 e^t + c_2 t e^t$ and $y(t) = c_2 e^t$. We can confirm this result by writing our system in matrix form and using our formula for the general solution when repeated eigenvalues occur of $\vec{x} = c_1 e^{\lambda t} \vec{v} + c_2 e^{\lambda t} (\vec{w} + t\vec{v})$ where $\vec{v}$ is an eigenvalue and $\vec{w}$ is a generalized eigenvalue.