## *Worksheet 11*

**TITLE** Geometry of First Order Systems of ODEs
**CURRENT READING** Blanchard, 2.2

**Homework Set #6 due Friday October 3** (* indicates EXTRA CREDIT)
**Section 2.2**: 7, 8, 11, 21* (EXPLAIN!), 24, 26. **Section 2.4**: 2, 5, 7, 8.
**Section 2.5**: 2, 3. **Chapter 2 Review**: 2, 3, 7, 12, 13 15, 16, 20, 30*.

**SUMMARY**
We will learn how to create the beautiful pictures which can result when one does quantitative analysis on systems of ODEs (phase portraits).

## 1. Vector Notation and Vector Fields

Let $\vec{x} = \begin{bmatrix} R(t) \\ F(t) \end{bmatrix}$ and $\frac{d\vec{x}}{dt} = \begin{bmatrix} R'(t) \\ F'(t) \end{bmatrix}$, the Lotka-Volterra equations can be re-written as:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} aR - bRF \\ cF + dRF \end{bmatrix} = \vec{P}(\vec{x}, t)$$

Note that the above $\vec{P}$ is a vector function of a vector input, in the case of Lotka-Volterra $P : \mathbb{R}^3 \to \mathbb{R}^2$

DEFINITION: **fixed point of linear system of ODEs**

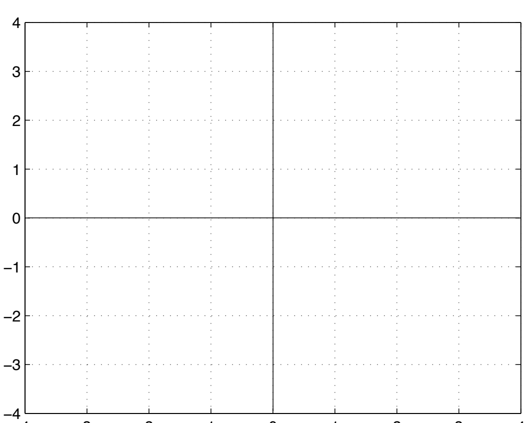
A fixed point or equilibrium point or stationary point $\vec{x_0}$ of the system $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, t)$ is a point at which $\vec{F}(\vec{x_0}) = \vec{0}$.

**RECALL**
We can visualize vector functions using **vector fields**. Consider the function $\vec{F}(x, y) = \begin{bmatrix} y \\ -x \end{bmatrix}$. Sketch the vector field in the axes to the left, below. Generally, we normalize the vectors to all have the same magnitude and produce something that is called a **direction field**. It looks exactly like a **slope field**, except the "lineal elements" have little arrows.

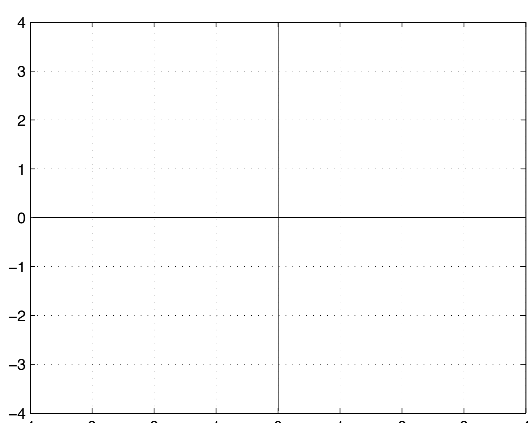
EXAMPLE

Let's draw the vector field and direction field for $\vec{F}(\vec{x}) = (y, -x)$.

**Vector Field** for $\vec{F} = (y, -x)$




**Direction Field** for $\vec{F} = (y, -x)$



## 2. Direction Fields and 1st Order Linear Systems of ODEs

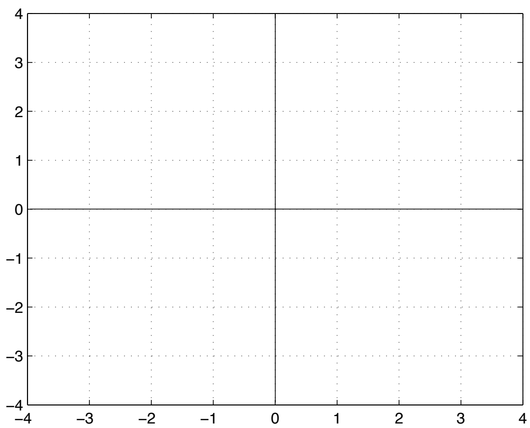
Consider the system of ODES

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x$$

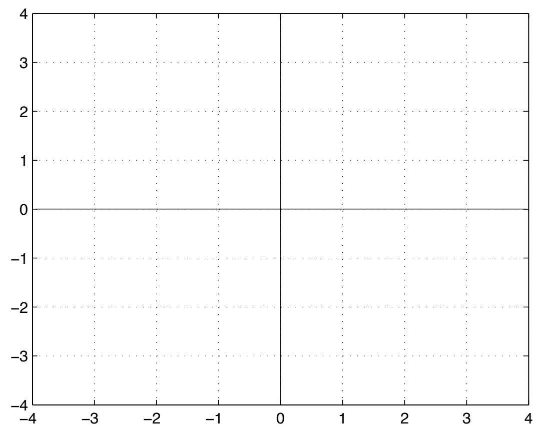
GROUPWORK

Draw a solution curve that starts at $x(0) = 0, \quad y(0) = 1$ on the axes to the left, and on the right draw solution curves of $x(t)$ and $y(t)$ on the same axes.

**Solution Curve** for $\vec{x}(0) = (0, 1)$




$x(t)$ and $y(t)$ versus $t$




**Q:** Does the system have any equilibria?
**A:** ____________________________________________

## 3. Nullclines

Consider the general 2-D system of ODEs

**SYSTEM A**

$$\frac{dx}{dt} = f(x, y) = y$$
$$\frac{dy}{dt} = g(x, y) = -x$$

DEFINITION: **nullcline**

A curve along which a derivative (with respect to the independent variable) is zero is said to be a nullcline. In other words, one of the variables will be constant, while the other variable varies with respect to $t$. An $x$-nullcline is a set of points for which $x$ is constant (i.e. $\frac{dx}{dt} = 0$). Algebraically, $\{(x, y) \mid \frac{dx}{dt} = 0\}$. A $y$-nullcline is a set of points for which $y$ is constant (i.e. $\frac{dy}{dt} = 0$). In this case, $\{(x, y) \mid \frac{dy}{dt} = 0\}$.

**Q:** What are the nullclines of System A?
**A:** ____________________________________________

## 4. Phase Portrait

The phase portrait of a system is a diagram showing the set of solution curves in the phase plane of a system of ODEs.

**SYSTEM B**

$$\frac{dx}{dt} = 5x$$
$$\frac{dy}{dt} = -y$$

**Exercise**

Sketch the phase portrait of the system. This means sketching the nullclines, clearly indicating any fixed points and including several solution curves in the phase plane.

**SYSTEM C**

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = 4x - 2y$$

**Exercise**

Sketch the phase portrait of the system. This means sketching the nullclines, clearly indicating any fixed points and including several solution curves in the phase plane.

**SYSTEM D**

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\frac{k}{m}x\end{aligned}$$

This system should look familar, or if it doesn't perhaps this differential equation is

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad (1)$$

The above equation is the equation of motion for **harmonic motion** and has the known solutions $x(t) = A\cos(\omega t) + B\sin(\omega t)$ where $\omega^2 = \dfrac{k}{m}$ and $\omega$ is the frequency of the motion and $\dfrac{2\pi}{\omega}$ is the period of the oscillation of a mass on a string (with no damping).

What's the relationship between Equation 1 and System D? Is there a way to convert one into the other?

GroupWork

Use technology to sketch phase portraits of System D for various values of the ratio $k/m$.