Differential Equations

Math 341 Fall 2014 — MWF 3:00-3:55pm Fowler 307

 — http://faculty.oxy.edu/ron/math/341/14/

Worksheet 8

TITLE Linear Differential Equations
CURRENT READING Blanchard, 1.8

Homework Set #5 due Friday September 26 (* indicates EXTRA CREDIT)
Section 1.9: 4, 5, 9, 12, 19, 22*.
Chapter 1 Review: 3, 4, 10, 11, 12, 13, 14, 26, 49, 52*.
Section 2.1: 1, 2, 3, 5, 7, 10, 14*.

SUMMARY
We will learn about the Linearity Principle and how it helps us to solve an entire class of ODEs. In addition we will learn some important new terms to describe ODEs, **homogeneous** and **non-homogeneous**. We'll try and make some analogies to some ideas already seen in Linear Algebra.

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1. Linearity Principles

DEFINITION: **first-order linear DE**

A **first-order linear DE** has the form $\dfrac{dy}{dt} = a(t)y + b(t)$. When $b(t) = 0$ for all t the equation is called **homogeneous**, otherwise the DE is called **non-homogeneous**.

The operator form of a nonhomogeneous first-order linear ODE is

$$\begin{aligned} \frac{dy}{dt} - a(t))y &= b(t)) \\ \left[\frac{d}{dt} - a(t)\right] y &= b(t) \\ \mathcal{L}y &= b(t) \end{aligned}$$

The operator $\mathcal{L}$ represents what actions are applied to the function $y(t)$. An operator is a mathematical object which takes a function as an input and output. Clearly, if $b(t) = 0$ then $\mathcal{L}y = 0$ and we name solutions of such equations as homogeneous solutions and denote them y_h. A solution which solves $\mathcal{L}y = b$ is denoted y_p and called a particular solution.

Differential Operators obey linearity properties reminiscent of Linear Transformations from Linear Algebra (Math 214). In other words, given any two function y_1 and y_2 and a constant $c \in \mathbb{R}$,

$$\mathcal{L}(y_1 + y_2) = \mathcal{L}y_1 + \mathcal{L}y_2 \text{ and } \mathcal{L}(cy) = c\mathcal{L}y$$

RECALL
A transformation (or mapping or function) $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$
2. $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v}$ in $\mathbb{R}^n$ and all scalars c.

THEOREM

If $y_h(t)$ is a solution of the homogeneous DE then so is $cy_h(t)$ where c is any real number.
If $y_1(t)$ and $y_2(t)$ are both solutions of the homogeneous DE, then so is $y_1(t) + y_2(t)$.

Exercise

Prove either one of the above statements, using operator notation.

THEOREM: **Extended Linearity Principles**

1) If $y_h(t)$ is any solution of the homogeneous DE and $y_p(t)$ is any solution of the non-homogeneous DE then $y_h + y_p$ is a solution of the non-homogeneous DE.
2) If $y_1(t)$ and $y_2(t)$ are solutions of the non-homogeneous DE, then $y_1(t) - y_2(t)$ is a solution of the associated homogeneous DE.

PROOF

Together we'll prove both of the above statements, using operator notation.

2. General Solution to a Linear ODE

The implication of the extended linearity principle is the idea that the general solution $y_g(t)$ to a non-homogeneous DE can be written as the sum of the general solution of the homogeneous DE $y_h(t)$ and one solution of the non-homogeneous DE $y_p(t)$. This is an incredibly important idea that repeats itself throughout the field of differential equations and other branches of mathematics.

This idea is very similar to the idea in Linear Algebra that the general solution to the linear system problem $A\vec{x} = \vec{b}$ is a vector $\vec{x}$ that can be written as a sum of two vectors, $\vec{x} = \vec{x}_h + \vec{x}_p$, where $\vec{x}_h$ is in the nullspace of A (i.e. solves the homogeneous problem $A\vec{x}_h = \vec{0}$) and $\vec{x}_p$ is in the rowspace of A (i.e. a vector which satisfies $A\vec{x}_p = \vec{b}$).

EXAMPLE

Let's try to find the general solution of the linear DE $\dfrac{dy}{dt} = -2y + e^t$.

GroupWork

Can we use the same technique to solve $\dfrac{dy}{dt} = -2y + 3e^{-2t}$? Let's try.

3. The Method of Undetermined Coefficients OR How To Guess Smartly
Given a linear DE of the form $\mathcal{L}y = f$ depending on what kind of function f is we can make various intelligent guesses as to what the particular form of the solution y_p will take.

We need to determine what "family of functions" the right hand side function belongs to, as this determines what our inttelligent guess will be. The Table below should give you an example of most functions that you are likely to run into.

Function	**Family**	**Guess**
k (a constant)	1	A
x^n (n integer)	Polynomial	$A + Bx + Cx^2 + \ldots Qx^n$
e^{kx} (k constant)	Exponential	Ae^{kx}
$\sin(kx)$ (k constant)	Trigonemtric	$A\sin(kx) + B\cos(kx)$
$\cos(kx)$ (k constant)	Trigonometric	$A\sin(kx) + B\cos(kx)$

EXAMPLE

Blanchard, page 123, #20.

What would a good guess for the solution to $\dfrac{dy}{dt} + 2y = 3t^2 + 2t - 1$ be?

Homework

Blanchard, page 122, #4.

Find the general solution of $\dfrac{dy}{dt} = 2y + \sin(2t)$.