Differential Equations

Math 341 Fall 2014 MWF 3:00-3:55pm Fowler 307

 http://faculty.oxy.edu/ron/math/341/14/

Class 1: Wednesday August 27

TITLE Introduction, Definitions & Terminology and Mathematical Models
CURRENT READING Blanchard, §1.1 and §1.2

Homework Assignment due Friday August 29
Section 1.1: 2,3,13,14.

SUMMARY
In today's class we shall go through various basic definitions and terminology associated with the study of differential equations in order to introduce you to the language we will be using throughout the course this semester. We'll also discuss ODEs as mathematical models.

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1. Definitions and Terminology

RECALL A function $y = f(x)$ is said to relate the dependent variable y and the independent variable x.

DEFINITION: **differential equation**
An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables is said to be a **differential equation**, or DE.

Differential Equations are classified by **type**, **order** and **linearity**.

TYPE
There are two main *types* of differential equation: "ordinary" and "partial." An **ordinary differential equation** (ODE) contains derivatives with respect to only one independent variable (though there may be multiple *dependent* variables). A **partial differential equation** contains partial derivatives with respect to multiple independent variables.

EXAMPLE
Consider the following differential equations: Classify them as either ODEs or PDEs.

$$(A)\frac{d^2u}{dx^2} + \lambda e^u = 0 \qquad (B)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (C)\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$$

RECALL: There are different notations to denote differentiation of a dependent variable y with respect to an independent variable x. **Leibniz Notation** has the form $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc. **Prime Notation** has the form $y', y'', y''', y^{(iv)}$ etc. **Dot Notation** is used (primarily by physicists and engineers) when the independent variable is t and has the form $\dot{y}, \ddot{y}$ etc. **Partial Differentiation** has the form u_x, u_{xy}, u_{xx} etc.

ORDER
The *order* of a differential equation is the order of the highest derivative found in the DE.
RECALL: The **order of a derivative** is the number of times the dependent variable is being differentiated with respect to the independent variable.

EXAMPLE

What is the order of each of the following differential equations?

(A) $\dfrac{d^2u}{dx^2} + \left(\dfrac{du}{dx}\right)^3 + 4u\sin(x) = 0$

(B) $\dfrac{du}{dx} - \left(\dfrac{u^2}{x^2+1}\right)^2 + \ln(x) = 0$

(C) $y''' - 2e^x y'' + 5\cos(x)y' = 20$

The most general form of an n-th order ordinary differential equation is $F(x, y, y', y'', y''', \ldots, y^{(n)}) = 0$ where F is a real-valued function of $n+2$ variables x, $y(x)$, $y'(x)$,...,$y^{(n)}(x)$.

The **normal form** of an n-th order differential equation involves solving for the highest derivative and placing all the other terms on the other side of the equation, i.e.

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

For example, during the class we shall grow very familiar with the normal form of first order ordinary differential equations, which look like: $y' = f(x, y)$.

Q: Write down the normal form of a second order ordinary differential equation here:
A:

LINEARITY

An n-th order differential equation is said to be **linear** if the function F is linear in the variables $y, y', y'', \ldots, y^{(n)}$. (HINT: what variable is missing from this list?) To be specific, an n-th order ODE will have the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

A **nonlinear** ODE is one that is not linear in the *dependent variable* (or its derivatives).

AUTONOMY

An **autonomous** ODE is one where the function F is an explicit function of the dependent variable only, i.e. $y' = F(y)$.
A **non-autonomous** ODE one where the function F contains the independent variable, i.e. $y'(x) = F(x, y)$.

GROUPWORK

Write down an example of one linear ODE and one nonlinear ODE. (Compare your answers with at least one of your neighbors.)

2. Solutions of ODEs

DEFINITION: **solution**

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n-th order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.(An interval is a continuous set of real numbers which may or may not contain the end points.)

In other words, a solution ϕ of an n-th order ODE is a function which possesses at least n derivatives and for which

$$F(x, \phi(x), \phi'(x), \phi''(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I$$

We say that ϕ *satisfies* (or solves) the differential equation on I.

The interval I in the above definition is also known as the **interval of definition**, **interval of existence**, **interval of validity** or the **domain of the solution** of the ordinary differential equation $F(x, y, y', y'', \ldots, y^{(n)}) = 0$.

EXAMPLE

Consider the ODE $xy' + y = 0$. Confirm that $y = 1/x$ is the solution of the ODE. What is the domain of definition of the function $y = 1/x$? What is the interval of definition of the solution of the ODE? Are these two sets identical?

DEFINITION: **solution curve**

A graph of the solution ϕ of an ODE is called a **solution curve**. As noted in the previous example, there may be a difference between the domain of definition of the function ϕ and the interval of definition of the solution ϕ. Thus there may be a difference between the graph of the function $\phi(x)$ and the solution ϕ on the interval of definition of the ODE.

DEFINITION: **initial value problem**

An **initial value problem** or IVP is a problem which consists of an n-th order ordinary differential equation combined with n initial conditions defined at a point x_0 found in the interval of definition I.

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \ldots, y^{(n-1)}) \qquad \text{differential equation}$$

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \ldots, y^{(n)}(x_0) = y_n \qquad \text{initial conditions}$$

where $y_0, y_1, y_2, \ldots, y_n$ are known constants.

For example, a first-order IVP looks like $y' = f(x, y), \quad y(x_0) = y_0$ and
a second-order IVP looks like $y'' = f(x, y, y'), \quad y(x_0) = y_0$ and $y'(x_0) = y_1$.

3. ODEs as Mathematical Models

DEFINITION: **mathematical model**

A **mathematical model** is a mathematical description of a system or phenomenon. Many physical systems often involve time (the variable t) so that the mathematical description of the model involves the rate of change of a variable with respect to time which can be mathematically represnted using differential equations. The solution of the model produces a state of the system at certain points in time: the past, present or future.

In Section 1.1 of the text, Blanchard, Devaney and Hall introduce a number of differential equations which serve as mathematical models of various physical systems and phenomena.

Often mathematical models begin with a relationship between the rate of change of a quantity and some other quantity. Initial conditions are often implied.

Malthusian Population Model

A mathematical model of human population growth was introduced by an English economist named Thomas Malthus in 1798. The main assumption is that the rate of growth of the population is proportional to the current population. Mathematically,

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$$

This model actually pretty accurately reflected the population growth of the United States in its early stages of development. Can you think of some drawbacks of the model?

Logistic Population Model of Verhulst

In 1846 a Belgian mathematician named Pierre Verhulst developed another model of human population growth. Its main assumption is that the relative rate of growth is proportional to the excess population an environment can hold. Instead of the relative rate of growth being constant, it decreases with the growth of the population itself. This model is know known as the **logistic model**. Mathematically,

$$\frac{dP}{dt}\frac{1}{P} \propto 1 - \frac{P}{M} \Rightarrow \frac{dP}{dt} = kP(1 - \frac{P}{M})$$

Differential equations are used to map all sorts of physical phenomena, from chemical reactions, disease progression, motions of objects, electronic circuits, et cetera. Most mathematical models of real-world situations do not have analytical solutions. For example, in 2001 I published a paper with a differential equation model that describes how movies make money over time (Edwards and Buckmire, "A Differential Equation Model of North American Cinematic Box-Office Dynamics," *IMA Journal of Management Mathematics*, **12**(1)). The model is represented by an initial value problem (IVP):

$$\begin{aligned}
\frac{dG}{dt} &= SA, \qquad G(0) = 0 \\
\frac{dS}{dt} &= -(S - A), \qquad S(0) = S_0 \\
\frac{dA}{dt} &= -\alpha\left(\frac{S}{S+\gamma} + \beta G\right)A, \qquad A(0) = A_0
\end{aligned}$$

Generally, the more complicated the physical phenomena being modelled is, the more complicated the equations descriving it will be.