Differential Equations

Math 341 Fall 2013 — MWF 12:50-1:45pm Fowler 307

 — `http://faculty.oxy.edu/ron/math/341/13/`

Worksheet 25

TITLE The Laplace Transform and The Heaviside Function
CURRENT READING Blanchard, 6.2

Homework Assignments due Monday November 18
Section 5.3: 2, 9, 12, 13, 14, 18*.
Chapter 5 Review: 1, 2, 6, 7, 8, 9, 11, 12, 26, 27*.
Section 6.1: 2, 3, 5, 7, 8, 9, 15, 18, 25*.
Section 6.2: 1, 2, 4, 8, 15, 16, 18*.
Homework Assignments due Monday November 25
Section 6.3: 5, 6, 8, 15, 18, 27, 28.
Section 6.4: 1, 2, 6, 7*.

SUMMARY
We shall continue our analysis of Laplace Transforms by considering discontinuous functions.

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1. Translation in t

DEFINITION: **Heaviside function**

The **unit step function** or **Heaviside function** $\mathcal{H}(t)$ is defined to be **0** when its argument is less than zero and **1** when its argument is greater than or equal to zero. Generally, it is written as $\mathcal{H}_a$ or $\mathcal{H}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

NOTE: **Blanchard, Devaney & Hall** uses the notation $u_a(t)$ for $\mathcal{H}(t-a)$.

Exercise

Sketch a picture of $u_a(t)$ in the space below. Is this function piecewise continuous? (Why do we care?) What is the definition of "piecewise continuous"?

Let's show that $\mathcal{L}[\mathcal{H}(t-a)] = \dfrac{e^{-as}}{s}$

$\boxed{\text{GROUPWORK}}$

Confirm that $f_1(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases}$
can be written as $f_1(t) = g(t) - g(t)\mathcal{H}(t-a) + h(t)\mathcal{H}(t-a)$ or $f_1(t) = g(t) + \mathcal{H}_a(t)(h(t) - g(t))$

How would you combine Heaviside functions to represent the following function? [HINT: what would the graph of the difference of two Heaviside functions look like?]

$$f_2(t) = \begin{cases} 0, & 0 \leq t < a \\ g(t), & a \leq t < b \\ 0, & t \geq b \end{cases}$$

This kind of function $f_2(t)$ is an example of an **interval function**, and is denoted $u_{ab}(t)$.
$u_{ab}(t) = 1$ if $a < t < b$ and 0 otherwise.

$\boxed{\text{EXAMPLE}}$

Blanchard, Devaney & Hall, page 586, #15. Suppose $a \geq 0$. Find the general solution of $\dfrac{dy}{dt} = -y + u_a(t)$

$\boxed{\text{THEOREM: } \textbf{Second Translation Theorem}}$

If $F(s) = \mathcal{L}[f(t)]$ and $a > 0$ is any positive real number, then $\mathcal{L}[f(t-a)\mathcal{H}(t-a)] = e^{-as}F(s)$.
It directly follows then that $\mathcal{L}[\mathcal{H}(t-a)] = \dfrac{e^{-as}}{s}$.

Corollary
$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\mathcal{H}(t-a)$

$\boxed{\text{THEOREM: } \textbf{Alternate form of the Second Translation Theorem}}$

It can be annoying to try and get the function which is multiplying the Heaviside function into the form $f(t-a)$ for use in the previous version of the Second Translation Theorem so a more useful results is: $\mathcal{L}[g(t)\mathcal{H}(t-a)] = e^{-as}\mathcal{L}[g(t+a)]$

2. Translation in s

THEOREM: **First Translation Theorem**

If $F(s) = \mathcal{L}[f(t)]$ and a is any real number, then $\mathcal{L}[e^{at}f(t)] = F(s-a)$. Sometimes the notation $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]|_{s\to s-a}$ is used.

Corollary

The inverse of the First Translation Theorem can be written as $\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t)$.

Exercise Given that $\dfrac{2s+5}{(s-3)^2} = \dfrac{2}{s-3} + \dfrac{11}{(s-3)^2}$, compute $\mathcal{L}^{-1}\left[\dfrac{2s+5}{(s-3)^2}\right]$. (HINT: recall that $\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] = t$)

EXAMPLE Compute $\mathcal{L}^{-1}\left[\dfrac{s/2+5/3}{s^2+4s+6}\right]$.

HINT: recall $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+k^2}\right] = \cos(kt)$ and $\mathcal{L}^{-1}\left[\dfrac{k}{s^2+k^2}\right] = \sin(kt)$

$\boxed{\text{EXAMPLE}}$ **Zill, Example 3, page 295.** Let's use Laplace Transforms to show that the solution of $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$ is $y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4 e^{3t}$.