# $\mathrm{D}$ifferential $\mathrm{E}$quations

## *Worksheet 22*

**TITLE** Hamiltonian Systems
**CURRENT READING** Blanchard, 5.1 & 5.3

**Homework Assignments due Monday November 11** (* indicates EXTRA CREDIT)
**Section 3.7**: 1,2*,6.
**Chapter 3 Review**: 3, 4, 6, 10, 13, 20*.
**Section 5.1**: 3, 4, 5, 8, 18, 20*.

**SUMMARY**
We shall continue our analysis of non-linear systems by introducing the concept of a Hamiltonian function.

Consider the following nonlinear planar system of ODEs

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^2$$

**Exercise**

Show that the function $H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$ has the property that $\dfrac{dH}{dt} = 0$ if $x$ and $y$ simultaneously satisfy the given system of ODEs. (HINT: Use the Differentiation Chain Rule!)

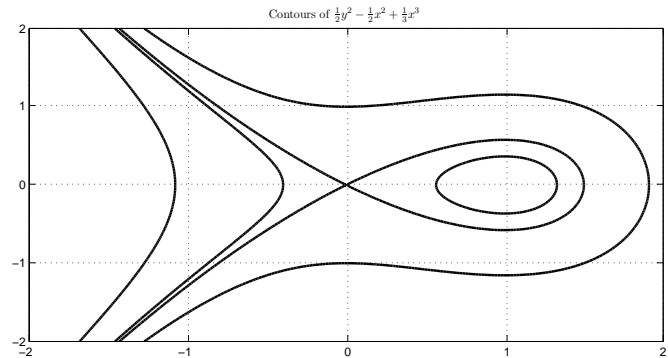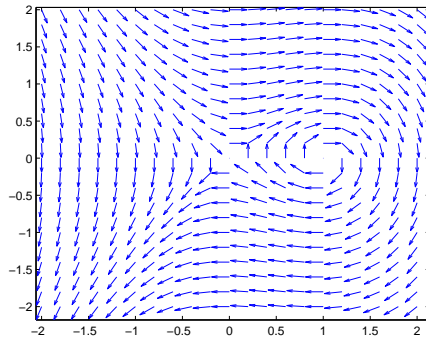## 1. The Hamiltonian

**DEFINITION: Hamiltonian function**

A real-valued function $H(x,y)$ is considered to be a conserved quantity for a system of ordinary differential equations if it is constant along ALL solution curves of the system. In other words, IF $(x(t), y(t))$ is a solution of the system then $H(x(t), y(t))$ is constant for all time which also implies that $\dfrac{d}{dt}H(x(t), y(t)) = 0$. The function $H(x,y)$ is known as the Hamiltonian function (or Hamiltonian) of the system of ODEs.

**2**. **The Hamiltonian Level Curves and The Phase Portrait**

**RECALL**

The **level curves** or **contours** of the function $H(x, y)$ are the set of points in the plane which satisfy the equation $H(x, y) = k$ for certain real values $k$.

Let's compare the level curves of $H(x, y) = \dfrac{1}{2}y^2 - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3$ with the direction field of the system $\dot{x} = y; \quad \dot{y} = x - x^2$. What do you notice?



Contours of $\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$

**3**. **Hamiltonian System**

**DEFINITION: Hamiltonian System**

A system of differential equations is called a **Hamiltonian system** if there exists a real-valued function $H(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

for all $x$ and $y$. The function $H$ is called the Hamiltonian function for the system.

**EXAMPLE**

The Hamiltonian often has a physical meaning for the sysem of ODEs that is modelling a partcular real-world situation, since it represents a quantity that is being conserved over time. For example, consider the system of ODEs that represents the **undamped** harmonic oscillator $y'' + qy = 0$:

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy$$

Let's show that the Hamiltonian for this system is $H(y, v) = \dfrac{1}{2}v^2 + \dfrac{q}{2}y^2$ which represents the total energy of the oscillator.

## 4. Obtaining Hamiltonians For Systems

In general the planar nonlinear system of first order DEs looks like

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

In order to find $H(x, y)$ we need to solve the following equations

$$f(x, y) = \frac{\partial H}{\partial y}$$
$$g(x, y) = -\frac{\partial H}{\partial x}$$

Does a Hamiltonian exist for this system? Well, if it does (and $H$ has continuous second partial derivatives) then $\dfrac{\partial^2 H}{\partial x \partial y} = \dfrac{\partial^2 H}{\partial y \partial x}$ which would mean that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} H_y = \frac{\partial}{\partial y} H_x = -\frac{\partial g}{\partial y}$$

So in order to check whether a given system of ODEs has a Hamiltonian or not all one needs to do is check whether

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$

Exercise
Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\frac{dx}{dt} = x + y^2$$
$$\frac{dy}{dt} = y^2 - x$$

EXAMPLE
Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\frac{dx}{dt} = -x \sin(y) + 2y$$
$$\frac{dy}{dt} = -\cos(y)$$

5. **Equilibria of Hamiltonian Systems**

**Hamiltonian Systems Can Never Have Sources or Sinks As Equilibria.** How could we prove that statement?

Consider

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

at the point $(x_0, y_0)$ which is the equilibrium point. Let's use the Linearization Technique!

The Jacobian of the linearized version of the Hamiltonian System at $(x_0, y_0)$ will be

What is the trace and determinant of the Jacobian matrix evaluated at $(x_0, y_0)$?

What are the eigenvalues of $J(x_0, y_0)$?

What do the eigenvalues allow us to conclude about behavior of the system with Hamiltonians near these fixed points?