
Differential Equations

Math 341 Fall 2013
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MWF 12:50-1:45pm Fowler 307
<http://faculty.oxy.edu/ron/math/341/13/>

Worksheet 21

TITLE Linearization: Analyzing Quasi-Linear Systems of ODEs

CURRENT READING Blanchard, 5.1

Homework Assignments due Monday November 11 (* indicates EXTRA CREDIT)

Section 3.7: 1,2*,6.

Chapter 3 Review: 3, 4, 6, 10, 13, 20*.

Section 5.1: 3, 4, 5, 8, 18, 20*.

Section 5.3: 2, 9, 12, 13, 14, 18*.

SUMMARY

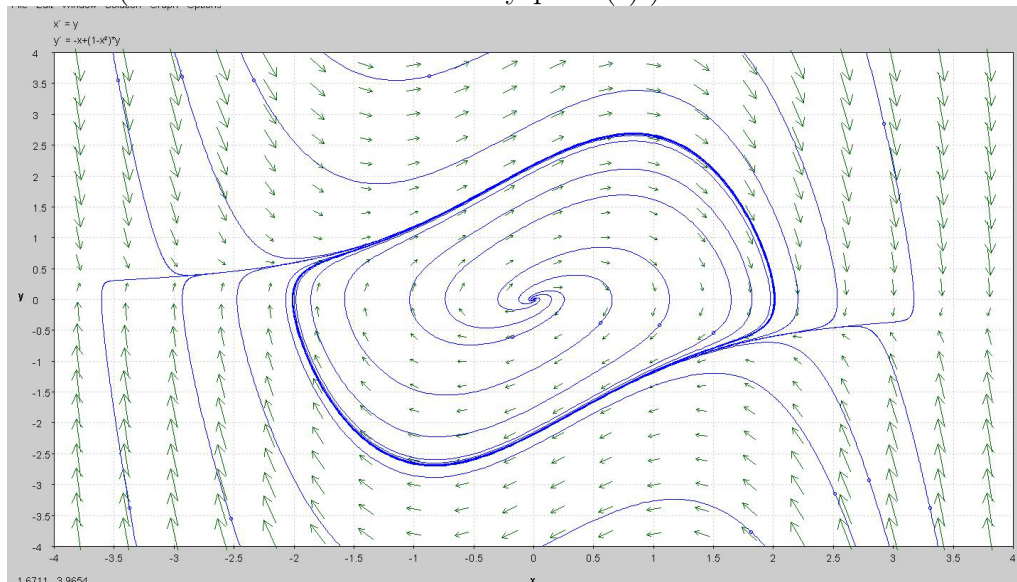
We shall begin our analysis of non-linear systems using a technique called linearization which transforms the behavior of nonlinear systems of ODEs back into our now familiar analysis of linear systems of ODEs. Remember your Taylor Approximations!

1. The Van der Pol Equation

An important nonlinear system of ODEs which occurs in Physics is the Van der Pol Equation for $x(t)$ $x'' + x - (1 - x^2)x' = 0$ which can be written as a non-linear system as

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + (1 - x^2)y\end{aligned}$$

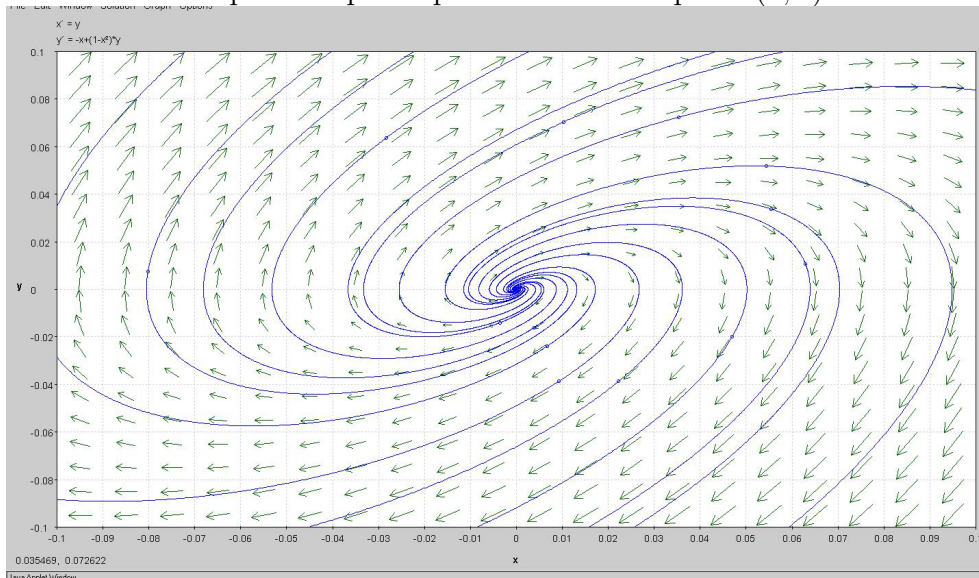
Below is the direction field and phase portrait for the Van der Pol system. What do you notice? (HINT: Locate the stationary point(s)!)



Q: What happens to solutions that start near the origin at (0,0)? What about solutions that start (relatively) far away at (3,3)?

A:

Here is a close up of the phase portrait near the point $(0,0)$



Q: What can we say about the stationary point at $(0,0)$ of the Van der Pol system?

A: _____

EXAMPLE

Let's use the technique of linearization to explain the behavior near the origin of the Van der Pol system. Suppose x and y are close to 0.1 in size, then the nonlinear term x^2y will be close to (0.1^3) in magnitude, much _____ then either x or y .

We can therefore write a linearized version of the Van der Pol system which looks like

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + y\end{aligned}$$

which when written as a matrix looks like $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Exercise

Find the eigenvalues of the linearized Van der Pol system and use this information to classify the stationary point at $(0,0)$.

2. The Linearization Process

RECALL

Definition: Jacobian matrix

The **derivative matrix** (usually called the **Jacobian**) of a vector function $\vec{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the matrix consisting of the n partial derivatives of each of the m co-ordinate functions arranged so that the rows of the matrix are exactly gradient vectors of each coordinate function. The Jacobian has mn entries where $J_{i,j} = \frac{\partial f_i}{\partial x_j}$. In other words,

$$J(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Consider the general form of a 2-dimensional nonlinear system of 1st order ODEs

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

This can also be thought of as $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$. Clearly in this case $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so the Jacobian matrix J for $\vec{f}(\vec{x})$ would be a _____.

We can always use Taylor's Theorem for Vector-Valued Functions to approximate the function $\vec{f}(\vec{x})$ near a point \vec{x}_0 by saying

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{x}_0) + J(\vec{x}_0)(\vec{x} - \vec{x}_0) + \dots$$

This will be extremely useful if the nonlinear system has a fixed point at the point (x_0, y_0) (also known as \vec{x}_0) because then we will be able to analyze a linear system of the form

$$\frac{d\vec{x}}{dt} = J(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

instead of the original nonlinear system

In fact, usually the change of variables $\vec{u} = \vec{x} - \vec{x}_0$ will be made and we will be analyzing the system

$$\frac{d\vec{u}}{dt} = J(x_0, y_0)\vec{u}, \quad \text{where } \vec{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

where the fixed point will now be at the origin of the (u, v) -system instead of at (x_0, y_0) in the xy -plane.

GROUPWORK

Consider

$$\begin{aligned}\frac{dx}{dt} &= -x + x^3 \\ \frac{dy}{dt} &= -2y\end{aligned}$$

Identify and then classify all the equilibria of the non-linear system of ODEs, using the Linearization Process. (HINT: calculate the Jacobian, evaluate at each equilibria, compute the eigenvalues and classify the equilibria)