

Differential Equations

Math 341 Fall 2013

MWF 12:50-1:45pm Fowler 307

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Worksheet 20

TITLE The Trace-Determinant Plane

CURRENT READING Blanchard, 3.7

Homework Assignments due Monday November 11 (* indicates EXTRA CREDIT)

Section 3.7: 1,2*,6.

Chapter 3 Review: 3, 4, 6, 10, 13, 20*.

Section 5.1: 3, 4, 5, 8, 18, 20*.

Section 5.3: 2, 9, 12, 13, 14, 18*.

SUMMARY

We shall summarize all the possible equilibria one can get with a 2x2 linear system of ODEs into one big picture!

1. Summarizing The Possibilities

Given a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the characteristic polynomial is $(a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$.

GROUPWORK

Your goal is to match the case # in the left column with the description of its critical point on the right (the list now is jumbled).

CASE 1: Real λ , $\lambda_1\lambda_2 < 0$

CASE 2: Real λ , $\lambda_1 \& \lambda_2 < 0$

CASE 3: Real λ , $\lambda_1 \& \lambda_2 > 0$

CASE 4: Real λ , $\lambda_1 = \lambda_2 > 0$

CASE 5: Real λ , $\lambda_1 = \lambda_2 < 0$

CASE 6: Complex λ , $\text{Re}(\lambda) > 0$

CASE 7: Complex λ , $\text{Re}(\lambda) < 0$

CASE 8: Complex λ , $\text{Re}(\lambda) = 0$

A Center

B Spiral Source

C (Stable) Node

D (Unstable) Node

E Saddle

F Spiral Sink

G Sink

H Source

Run the `LinearPhasePortraits` program from the Blanchard, Devaney & Hall textbook software. Use the slide bars to obtain different values of a , b , c and d and the different kinds of eigenvalues recorded above in the Cases. Record your results in the table below.

CASE #	a	b	c	d	λ_1	λ_2	Description
1							
2							
3							
4							
5							
6							
7							
8							

For more details, see the handout from **Edwards and Penney**, *Differential Equations*, **3rd Edition**, Prentice Hall: 2004, pp 381-389.

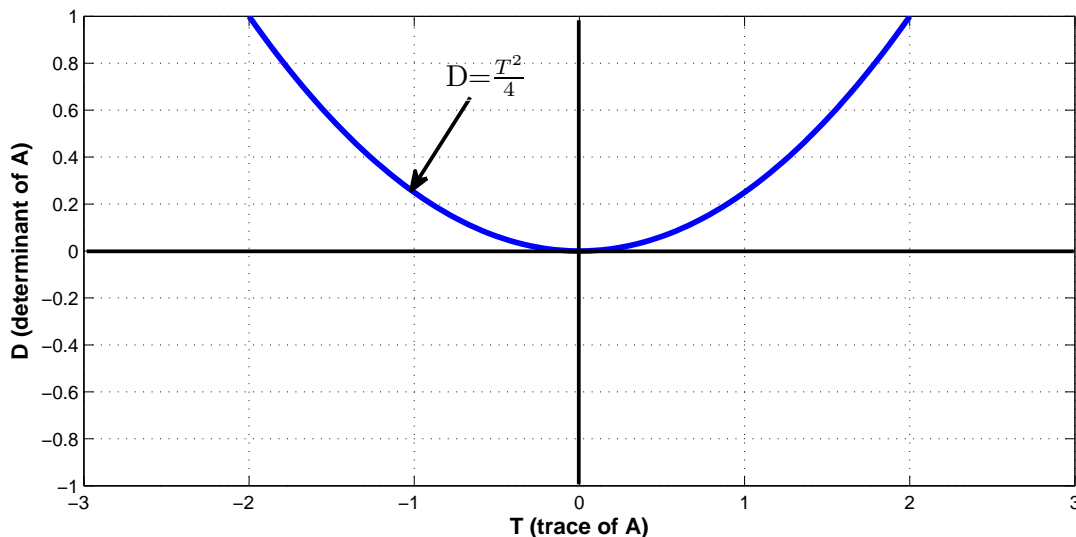
2. The Trace-Determinant Plane

Recall that the eigenvalues of a 2×2 matrix are given by the roots of the polynomial $p(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$.

It's also true that the trace of A , denoted $\text{tr}(A)$ is equal to the **sum** of the eigenvalues $\lambda_1 + \lambda_2$. Let's use the symbol T for $\text{tr}(A)$. The determinant of A , denoted $\det(A)$ is equal to the **product** of the eigenvalues $\lambda_1 \lambda_2$. Let's use the symbol D for $\det(A)$.

Then we know that the eigenvalues are given by the solutions to $\lambda^2 - T\lambda + D = 0$, or
$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

In other words, the condition on whether we will have real, complex or repeated eigenvalues depends on the behavior of the discriminant $\Gamma = T^2 - 4D$. See the figure drawn below. This is known as the **Trace-Determinant Plane**



This graph is an example of a parameter plane. As the matrix A changes it has different values of T and D and the linear system $\frac{d\vec{x}}{dt} = A\vec{x}$ corresponding to that matrix will be located at a different location in (T, D) -space.

Exercise

- (1) What kind of phase portraits will exist in (T, D) -space along the D axis?
- (2) What about the T -axis?
- (3) What kind of phase portraits occur along the curve $D = \frac{T^2}{4}$?
- (4) What happens as one moves from the region just above the T -axis ($D > 0$) to just below the T -axis ($D < 0$)? Does it matter if $T > 0$ or $T < 0$?
- (5) What kinds of solutions exist in the region above the parabola $D = \frac{T^2}{4}$?