Test 2: Differential Equations

Math 341 Fall 2013 | Friday November 22
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Directions: Read *all* problems first before answering any of them. There are six (6) pages in this test. This is a 55-minute, limited-notes*, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

***You may use a one-sided 8.5" by 11" "cheat sheet" which must be stapled to the exam.**

Offer: If there is a formula or piece of information that you feel that you need in order to solve a problem, I will provide it to you at a non-negotiable rate of at least a one point deduction.

Pledge: I, ________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		15
2		15
3		20
BONUS		5
Total		**50**

1. *[15 points total.]* **Linear Systems of Differential Equations, Trace-Determinant Plane, Bifurcation.** VISUAL & ANALYTIC & VERBAL.

Consider $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}$ and α is a known real-valued parameter. Recall that the eigenvalues of the matrix A are given by $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$ where T is the trace of the matrix A and D is the determinant of matrix A.

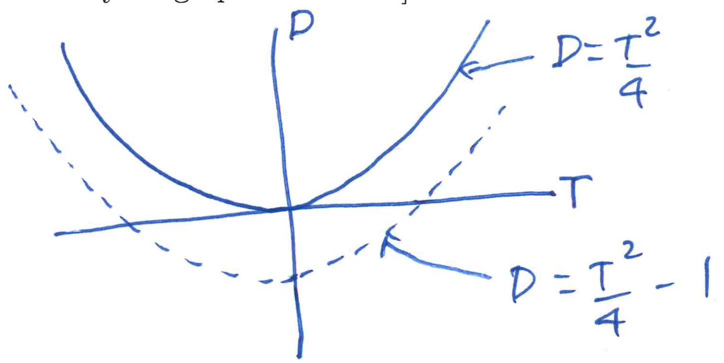
1(a) *[5 points].* Compute the trace T and determinant D of the matrix A in terms of the values of α. Show that the algebraic relationship between the trace T and the determinant D for the matrix A is $T^2 - 4D - 4 = 0$ for all values of α.

$T = 2\alpha$

$D = \alpha^2 - 1$

$D = \left(\frac{T}{2}\right)^2 - 1 \quad \Rightarrow D = \frac{T^2}{4} - 1 \Rightarrow T^2 - 4D - 4 = 0$

1(b) *[5 points].* Sketch a graph in the trace-determinant plane depicting the relationship between the trace T and determinant D for the given matrix A as α changes. On the same axes, sketch the standard graph in the trace-determinant plane which separates the occurrence of real eigenvalues from complex eigenvalues for *any* matrix. [HINT: Label each of your graphs and axes!]



1(c) *[5 points].* Explain why the system of differential equations $\frac{d\vec{x}}{dt} = A\vec{x}$ corresponding to $A = \begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}$ will never have periodic solutions (i.e. spirals or centers) regardless of the value α.

For periodic solution you need complex eigenvalues, for this problem $T^2 - 4D = 4$

$\lambda = \frac{2\alpha \pm \sqrt{4}}{2} = \alpha \pm 1$ which is always real, so no periodic sol'n (spirals or centers) are possible

2. *[15 points total.]* **Linearization, Hamiltonian function, Gradient function, Jacobian, Lyapunov function.** ANALYTIC, VERBAL & VISUAL.
Determine whether the following statements are **TRUE** or **FALSE** and place your answer in the box. To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! The explanation for your answer is worth FOUR POINTS while your TRUE or FALSE answer is worth 1 point.

Consider the quasi-linear system below (β is a known real-valued parameter):

$$\frac{dx}{dt} = \beta y + e^{x+y} = f(x,y)$$
$$\frac{dy}{dt} = \beta x + e^{x+y} = g(x,y)$$

2(a) TRUE or FALSE? "The function $H(x, y) = \beta xy + e^{x+y}$ is a Hamiltonian function for the given nonlinear system of ODEs."

$\boxed{\text{FALSE}}$

Hamiltonian condition $\frac{dH}{dt} = 0$

$f_x = -g_y$

$f_x = e^{x+y} \cdot 1$

$g_y = e^{x+y} \cdot 1$

$e^{x+y} = -e^{x+y}$ never!

so, no Hamiltonian exists

2(b) TRUE or FALSE? "The function $G(x, y) = \beta xy + e^{x+y}$ is a Gradient function for the given nonlinear system of ODEs."

$\boxed{\text{TRUE}}$

Gradient Condition $\dot{\vec{x}} = \nabla G$

$f_y = \beta + e^{x+y}$

$g_x = \beta + e^{x+y}$

$f_y = g_x$

$e^{x+y} + \beta = \beta + e^{x+y}$

gradient exists!

2(c) TRUE or FALSE? "The function $L(x, y) = -\beta xy - e^{x+y}$ is a Lyapunov function for the given nonlinear system of ODEs."

$\boxed{\text{TRUE}}$

$L = -G$ is a Lyapunov function if G is a gradient function

$$\frac{dL}{dt} = (-G)_x \frac{dx}{dt} + (-G)_y \frac{dy}{dt} = L_x \dot{x} + L_y \dot{y}$$

~~$= -(e^{x+y} \cdot (\beta y + e^{x+y})$~~

$= (-\beta y - e^{x+y})(\beta y + e^{x+y}) + (\beta x + e^{x+y})(\beta x + e^{x+y})$

$= -(\beta y + e^{x+y})^2 + (\beta x + e^{x+y})^2 < 0 \ \forall x, y$

3. *[20 points total.]* **Laplace Transforms, Partial Fractions, Eigenvector, Eigenvalues, Phase Potraits.** ANALYTIC & VERBAL.
Consider the BONUS question from 2013 Exam 1:

$$y'' + 3y' + 2y = 0 \text{ where } y(0) = 1, \quad y'(0) = 0.$$

3(a) *[5 points].* Show that the Laplace Transform $Y(s)$ of the exact solution $y(t)$ to this problem is $Y(s) = \dfrac{s+3}{(s+2)(s+1)}$.

$$\mathcal{L}\{y''\} + \mathcal{L}\{3y'\} + \mathcal{L}\{2y\} = \mathcal{L}\{0\}$$

$$s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = 0$$

$$s^2Y - s - 0 + 3sY - 3 + 2Y = 0$$

$$(s^2 + 3s + 2)Y - s - 3 = 0$$

$$(s^2 + 3s + 2)Y = s + 3$$

$$Y = \frac{s+3}{(s+2)(s+1)}$$

3(b) *[5 points].* Use the result in 3(a) to show that the exact solution to $y'' + 3y' + 2y = 0$ where $y(0) = 1, y'(0) = 0$ is $y(t) = 2e^{-t} - e^{-2t}$.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s+3}{(s+2)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\}$$

$$= -e^{-2t} + 2e^{-t}$$

$$\frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} = \frac{-1}{s+2} + \frac{2}{s+1}$$

$$s + 3 = A(s+1) + B(s+2)$$

$s = -1$, $\quad 2 = B \cdot 1$

$s = -2$ $\quad 1 = A(-1)$

3(c) *[5 points]*. Rewrite the original second order differential equation $y'' + 3y' + 2y = 0$ into a two-dimensional system of first order differential equations $\frac{d\vec{x}}{dt} = A\vec{x}$ where $\vec{x} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ with initial condition $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Write down the exact solution $\vec{x}(t)$ to this initial value problem in the form $c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$, where $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$. [HINT: You should not have to *compute* any eigenvectors or eigenvalues for this problem.]

$y = 2e^{-t} - e^{-2t}$

$y' = -2e^{-t} + 2e^{-2t}$

$\vec{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t}$

$y'' = -3y' - 2y$

$(u)' = -3u - 2y$

$(y)' = u$

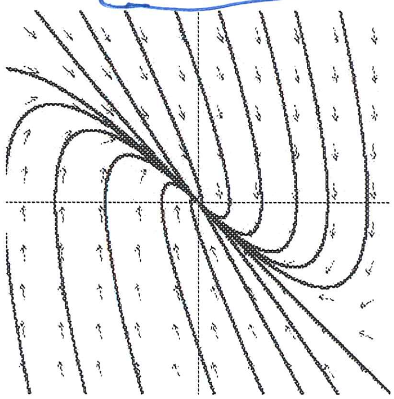
$$\frac{d}{dt}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -1\begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad E_{-1} = \text{span}\begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad E_{-2} = \text{span}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

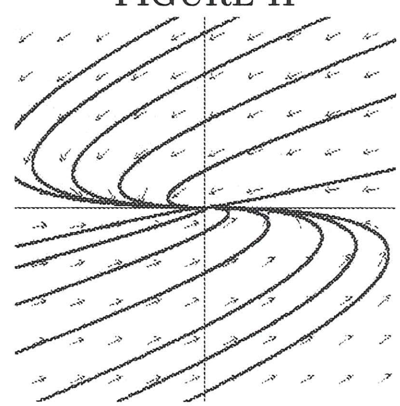
3(d) *[5 points]*. Which of the following figures contains a phase portrait that belongs to the 2-dimensional linear system of first order differential equations discussed in 3(c)? EXPLAIN YOUR ANSWER. What information do you use to make your choice of figure?

FIGURE I



As $t \to -\infty$, Solutions should look like $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ which is a ~~slope~~ negatively sloped line from 2nd to 4th quadrants which indicates FIGURE 1

FIGURE II



Also as $t \to \infty$, e^{-t} is dominant (goes to zero SLOWER than e^{-2t}) so $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ direction should appear, which is a line $y = -x$

BONUS. *[5 points].* Show that the Inverse Laplace Transform of $F(s) = \dfrac{e^{-as}}{s^2}$ is $f(t) = (t-a)\mathcal{H}(t-a)$, i.e. $\mathcal{L}\{(t-a)\mathcal{H}(t-a)\} = \dfrac{e^{-as}}{s^2}$. Sketch the graph of $f(t)$, assuming $a > 0$.

HARD WAY

$$\mathcal{L}\{(t-a)\mathcal{H}(t-a)\} = \int_0^\infty (t-a)\mathcal{H}(t-a)e^{-st}\,dt$$

$$= \int_a^\infty (t-a)e^{-st}\,dt$$

$$= \int_a^\infty te^{-st} - \int_a^\infty ae^{-st}\,dt$$

$u = t \quad du = dt$

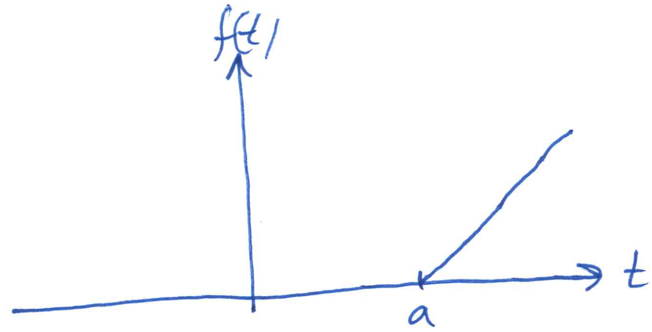
$dv = e^{-st}dt \quad v = \dfrac{e^{-st}}{-s}$

$$= \frac{te^{-st}}{-s}\Big|_a^\infty + \frac{1}{s}\int_a^\infty e^{-st}\,dt - \int_a^\infty ae^{-st}\,dt$$

$$= \frac{ae^{-sa}}{s} + \frac{1}{s}\left(\frac{1}{-s}e^{-st}\right)\Big|_a^\infty - \left(\frac{ae^{-st}}{-s}\right)\Big|_a^\infty$$

$$= \cancel{\frac{ae^{-sa}}{s}} + \frac{1}{s^2}e^{-as} - \frac{1}{s^2}\cancel{\lim_{b\to\infty} e^{-sb}}^{\,0} - \cancel{\frac{ae^{-sa}}{s}} + \cancel{\lim_{b\to\infty}\frac{ae^{-sb}}{s}}^{\,0} = \boxed{\frac{e^{-as}}{s^2}}$$

SKETCH



$$f(t) = \begin{cases} 0, & t < a \\ t-a, & t \geq a \end{cases}$$

EASY WAY

Second Translation Theorem

We know if $F(s) = \dfrac{1}{s^2}$ $\quad f(t) = t$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{e^{-as}\cdot\frac{1}{s^2}\right\} = (t-a)\cdot\mathcal{H}(t-a)$$