Test 1: Differential Equations

Math 341 Fall 2010 | Friday October 15

 | 2:30pm-3:25pm

Name: BUCKMIRE

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		20
2		15
3		15
BONUS		5
Total		**50**

1. *[20 points total.]* **Existence and Uniqueness Theorem, Functions, Equilibrium Solutions, Separation of Variables, Interval of Validity.** ANALYTIC & VERBAL. Consider the initial value problem

$$\frac{dy}{dt} = y^2, \quad y(0) = A, \text{where } A \text{ is any positive real number.}$$

1(a) *[5 points].* Assuming $A > 0$, show that the solution to this initial value problem has the form $y(t) = \dfrac{A}{1 - At}$. SHOW ALL YOUR WORK.

Check IC: $t = 0 \Rightarrow y = A$

$t = 0, \ y(0) = \dfrac{A}{1 - A \cdot 0} = A$ ✓

Check DE: $\dfrac{dy}{dt} = y^2$

$$\frac{d}{dt}\left(\frac{A}{1-At}\right) = \left(\frac{A}{1-At}\right)^2$$

$$A \cdot \frac{-1}{(1-At)^2} \cdot -A =$$

$$\frac{A^2}{(1-At)^2} = \frac{A^2}{(1-At)^2} \checkmark$$

1(b) *[5 points].* Assuming $A > 0$, what does the Existence and Uniqueness Theorem allow you to conclude about whether the solution in part **(a)** is the only solution to the given initial value problem? EXPLAIN YOUR ANSWER.

EUT says given, $y' = f(t, y), \ y(a) = b$, $f(t,y)$ continuous at $(a,b) \Rightarrow y(t)$ exists

$f_y(t,y)$ continuous at $(a,b) \Rightarrow y(t)$ unique

$f(t,y) = y^2$

$\dfrac{\partial f}{\partial y} = 2y$

Both y^2 and $2y$ are polynomials so they are continuous for all t and y so at $(0, A)$ we Know $y' = y^2$ has unique solution.

1(c) *[5 points]*. Assuming $A > 0$, what is the interval of validity $\mathcal{I}$ (i.e. the connected interval on the real line for which the solution in part **(a)** is valid)? Explain why *or if* the interval of validity $\mathcal{I}$ is different from the domain of the solution function given in part **(a)**. EXPLAIN YOUR ANSWER.

Domain of $y(t) = \frac{A}{1-At}$ is all $t \in \mathbb{R}$ except where $1 - At = 0$ i.e. $At = 1$, $t = 1/A$

Domain $= \{t \in \mathbb{R} \setminus \{1/A\}\} = \{t \in \mathbb{R}, t \neq \frac{1}{A}\}$

Interval of Validity must be $t > \frac{1}{A}$ or $t < \frac{1}{A}$

Since initial condition occurs at $t = 0$, it must be in $\mathcal{I}$. $0 < 1/A$ since $A > 0$.

Thus $\mathcal{I} = \{t \in \mathbb{R} \ni t < \frac{1}{A}\} = -\infty < t < \frac{1}{A}$

1(d) *[5 points]*. How does the interval of validity $\mathcal{I}$ from part **(c)** depend on the value of A? Consider $\lim_{A \to 0^+} \mathcal{I}$, i.e. let A get smaller and smaller while still remaining positive. Does the interval of validity $\mathcal{I}$ get larger or smaller as A gets closer and closer to zero? After taking the limit, what is the solution of the initial value problem $y' = y^2$, $y(0) = 0$ and its interval of validity? EXPLAIN YOUR ANSWER.

As $A \to 0^+$ the interval $\mathcal{I} = -\infty < t < \frac{1}{A}$ gets larger and larger as $\frac{1}{A} \to +\infty$.

When $A = 0$, $\mathcal{I} = -\infty < t < \infty$ or $t \in \mathbb{R}$ and in that case $y(t) = 0$ is the unique solution.

2. *[15 points total.]* **Slope Fields, Systems of DEs, Equilibria, Homogeneous and Non-homogeneous DEs.** ANALYTIC, VERBAL & VISUAL.
Short Answer Questions With Required Explanation. These questions are similar to reading quiz questions where you answer the question and then explain your answer in complete sentences. Much more credit is given for the EXPLANATION than the ANSWER to the question.

2(a) Is a **coupled system** of first order differential equations generally easier or harder to solve than a **decoupled system** of first order differential equations? EXPLAIN YOUR ANSWER.

Coupled systems are harder to solve because D.E. of one dependent variable depends on values of another dependent variable. You have to solve all equations simultaneously.
In decoupled systems you can solve them one equation at a time, since they do not affect other equations in the system

2(b) TRUE or FALSE: "If each of $y_1(t)$ and $y_2(t)$ are solutions to a **non-homogeneous, linear, first-order, ordinary differential equation**, then their sum $y_1(t) + y_2(t)$ must also be a solution to the same equation." Either prove the statement is TRUE or provide a counter-example and prove your counter-example shows the statement is FALSE in that case.

Let $Ly = b \neq 0$ be the non-homo DE.

$\mathcal{L}y_1 = b$

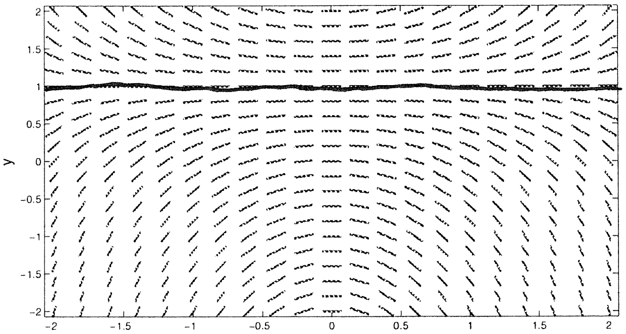
$\mathcal{L}y_2 = b$

$$\mathcal{L}(y_1+y_2) = \mathcal{L}y_1 + \mathcal{L}y_2 \quad \text{(linearity of } \mathcal{L}\text{)}$$
$$= b + b$$
$$= 2b \neq b \quad (\text{unless } b = 0)$$

But $\mathcal{L}(y_1+y_2) = b$ if y_1+y_2 is a solution to $\mathcal{L}y = b$

2(c) Consider the slope field for $y' = f(x, y)$ in the xy-plane shown in Figure 1 below. Does the unknown differential equation $y' = f(x, y)$ have any **equilibrium solutions**? How can you tell? Also, classify the unknown differential equation as either **autonomous or non-autonomous**. EXPLAIN YOUR ANSWER(S).

FIGURE 1



$y = 1$ looks like an equilibrium solution to DE. ~~It~~ The DE depends on x and y since slope elements change along $x=c$ (vertically) and $y=c$ (horizontally)

3. *[15 pts. total]* **Phase Lines, Equilibria, Bifurcations, Geometric Representations.** VISUAL & ANALYTIC.

Consider the following differential equation: $\frac{dy}{dt} = |y| + \alpha$, where α is a real-valued parameter.

3(a) *[5 points]* What are the equilibrium values of the differential equation? Identify them as y^*. They should depend on values of the parameter α.

$\frac{dy}{dt} = 0 = |y^*| + \alpha$ for equilibrium values

$\Rightarrow |y^*| = -\alpha$

When $\alpha > 0$ there are no solutions

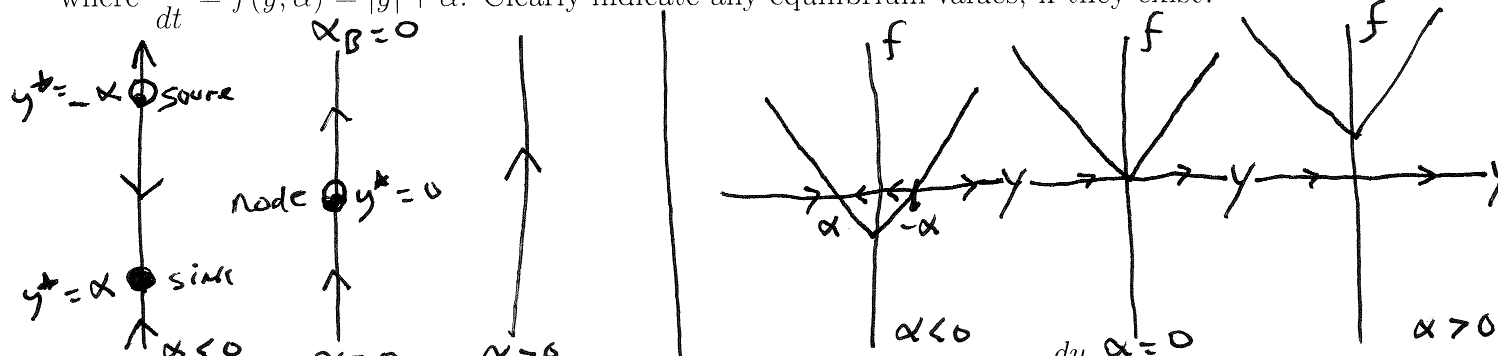
when $\alpha \le 0$, $y^* = \pm\alpha$ are equilibrium values.

ONLY DO ONE OF THE FOLLOWING TWO QUESTIONS LABELLED 3(b)

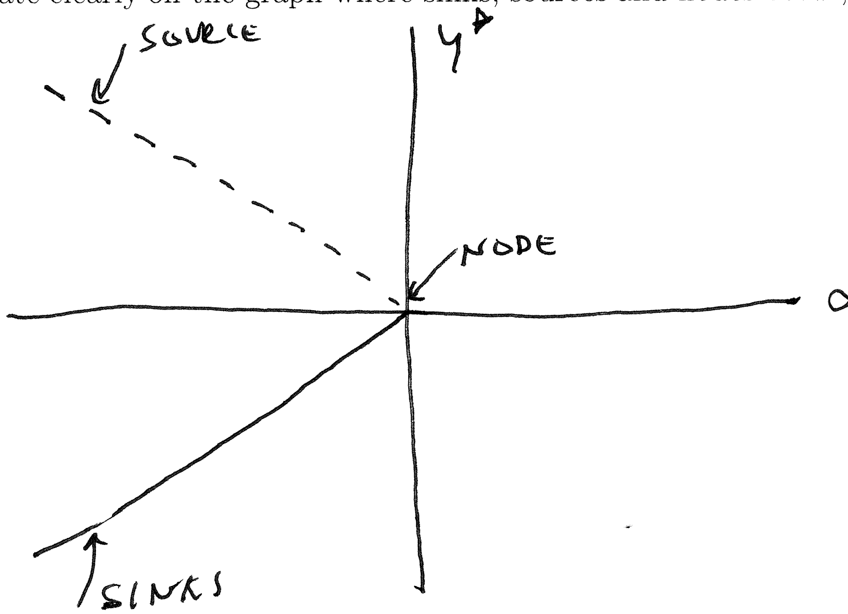
3(b) *[5 points]* Is there a bifurcation value for the parameter α? If so, call it α_B and draw phase lines corresponding to the cases where $\alpha < \alpha_B$, $\alpha = \alpha_B$ and $\alpha > \alpha_B$. Indicate locations (and values) of y^* on your phase lines.

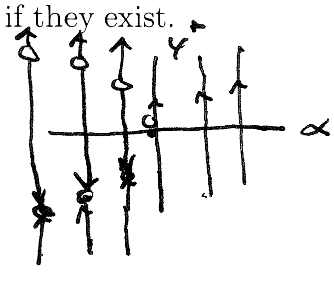
OR

3(b) *[5 points]* Is there a bifurcation value for the parameter α? If so, call it α_B and sketch graphs of $f(y;\alpha)$ versus y corresponding to the cases where $\alpha < \alpha_B$, $\alpha = \alpha_B$ and $\alpha > \alpha_B$ where $\frac{dy}{dt} = f(y;\alpha) = |y| + \alpha$. Clearly indicate any equilibrium values, if they exist.



3(c) *[5 points]* Draw a bifurcation diagram for the differential equation $\frac{dy}{dt} = |y| + \alpha$ in the αy^*-plane. Indicate clearly on the graph where sinks, sources and nodes occur, if they exist.





BONUS. *[5 points]* Solve $\frac{dy}{dx} + xy = x, y(0) = 0$.

Sep. of Vars.

$$\frac{dy}{dx} = x - xy = x(1-y)$$

$$\frac{dy}{1-y} = x\,dx$$

$$\int \frac{dy}{1-y} = \int x\,dx$$

$$-\ln|1-y| = \frac{x^2}{2} + C$$

$$\ln|1-y| = -\frac{x^2}{2} - C$$

$x=0, y=0$

$$\ln 1 = 0 - C \Rightarrow C = 0$$

$$\ln|1-y| = -\frac{x^2}{2}$$

$$1-y = e^{-x^2/2}$$

$$\boxed{1 - e^{-x^2/2} = y(x)}$$

Integrating Factor

$$\mu = e^{\int x\,dx} = e^{x^2/2}$$

$$y' e^{x^2/2} + x e^{x^2/2} y = x e^{x^2/2}$$

$$(y e^{x^2/2})' = x e^{x^2/2}$$

$$y e^{x^2/2} = \int x e^{x^2/2}\,dx$$

$$= e^{x^2/2} + C$$

$$y = 1 + Ce^{-x^2/2}$$

$x=0, y=0$

$$0 = 1 + C$$

$$-1 = C$$

$$\boxed{y(x) = 1 - e^{-x^2/2}}$$