Differential Equations

Worksheet 29: **Wednesday December 1**

TITLE Laplace Transforms and Introduction to Convolution
CURRENT READING Blanchard, 6.5

Homework Assignments due Monday December 6
Section 6.3: 5, 6, 8, 9, 10, 15, 18, 27, 28.
Section 6.4: 1, 2, 5, 7, 8.
Section 6.5: 2, 5, 6, 9

SUMMARY
We shall discuss the equivalent of the product rule for Laplace Transforms and be introduced to the concept of the convolution of two functions.

1. **Product Rule for Laplace Transforms**

DEFINITION: **convolution**

If two functions $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ then **the convolution of f and g**, usually denoted $f * g$ is defined to be

$$\int_0^t f(\tau)g(t - \tau)d\tau$$

NOTE: this "product" is a function of t. The use of the "*" symbol is deliberate, since the convolution operation has the following familiar properties:

THEOREM: properties of convolution

If f, g and h are piecewise continuous on $[0, \infty)$, then
I. $f * g = g * f$ (Commutative)
II. $f * (g + h) = f * g + f * h$ (Distributive Under Addition)
III. $f * (g * h) = (f * g) * h$ (Associative)
IV. $f * 0 = 0$ (Existence of Zero Object)

THEOREM: **The convolution theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order so that $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$ then $\mathcal{L}[f * g] = F(s)G(s)$.
Corollary
$\mathcal{L}^{-1}[F(s)G(s)] = f * g$.

Exercise Evaluate $\mathcal{L}\left[\int_0^t e^\tau \sin(t - \tau)d\tau\right]$. [HINT: use the convolution theorem!]

EXAMPLE

The convolution theorem allows us to find inverse Laplace Transforms without resorting to partial fractions.

For example, show that $\mathcal{L}^{-1}\left[\dfrac{k}{s^4 + k^2 s^2}\right] = \dfrac{kt - \sin(kt)}{k^2}$ by using the Convolution Theorem AND by using Partial Fractions.

Using Partial Fractions

Using Convolution

2. The General Solution To A Non-Homogeneous Linear Second Order ODE

Consider $y'' + py' + qy = f(t)$ with $y(0) = 0$ with $y'(0) = 0$. It can be shown that the exact solution $y(t)$ (for $t > 0$) to this problem is given by a convolution, namely $\xi(t) * f(t)$ where

$$\mathcal{L}[\xi(t)] = \frac{1}{s^2 + ps + q}$$

This is a pretty incredible result. It is known as **Duhamel's Principle**.

What one does is show that the $y = \xi(t)$ is the solution to the **unit impulse** version of the problem, i.e. $y'' + py' + qy = \delta_0(t)$ with $y(0) = 0$ with $y'(0) = 0$. Thus, if one wants to solve any other non-homogeneous problem all one needs to do is solve the unit impulse problem, and **convolve** that solution with the given non-homogeneous function.

$\boxed{\text{EXAMPLE}}$

Show that the general solution to $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0$ is

$$y(t) = \int_0^t \sin(t - u) \, f(u) \, du$$

3. **Derivatives of Laplace Transforms**

EXAMPLE

Show that $\dfrac{d}{ds}F(s) = -\mathcal{L}[tf(t)]$ and $\dfrac{d^2}{ds^2}F(s) = \mathcal{L}[t^2 f(t)]$.

THEOREM: Derivatives of Laplace Transforms

When $F(s) = \mathcal{L}[f(t)]$, and $n = 0, 1, 2, \ldots$ $\mathcal{L}[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}F(s)$

Exercise We now have TWO different ways to show that $\mathcal{L}[te^{at}] = \dfrac{1}{(s-a)^2}$ WITHOUT EVALUATING AN INTEGRAL. Do this.

4. **Laplace Transform of an Integral**

We can use the Convolution Theorem with $g(t) = 1$ and show that $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \dfrac{F(s)}{s}$

NOTE

I. Multiplication of $f(t)$ by t generally involves **differentiation** of its Laplace Transform $F(s)$ with respect to s.

II. Division of $F(s)$ by s generally involves **anti-differentiation** of its Inverse Laplace Transform $f(t)$ with respect to t

Mathematically, these statements can be expressed as

I. $\mathcal{L}[tf(t)] = \dfrac{d}{ds}(\mathcal{L}[f(t)])$ and **II.** $\mathcal{L}[f(t)] = \dfrac{\mathcal{L}[f'(t)]}{s}$ and are always true if the Function-Transform pair $f(t) \leftrightarrow F(s)$ has the property that $f(0) = 0$.

GROUPWORK

Find $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2 + 1)}\right]$

5. Volterra Integral Equations

A **Volterra integral equation** or **integro-differential equation** is an equation where the unknown function $f(t)$ (and/or $f'(t)$) appears on one side of the equation *and* in an integral on the other side, i.e.

$$f(t) = g(t) + \int_0^t f(\tau)h(t - \tau)d\tau$$

These kinds of problems show up in many different applications, often involving biological systems.

EXAMPLE

Zill, page 309, Example 5. Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau$

Exercise **Zill, page 313, HW #46**. Solve $y'(t) + 6y(t) + 9 \int_0^t y(\tau)d\tau = 1, \quad y(0) = 0$