Differential Equations

Math 341 Fall 2010 | MWF 2:30-3:25pm Fowler 307

 | `http://faculty.oxy.edu/ron/math/341/10/`

Worksheet 23: **Monday November 8**

TITLE Hamiltonian Systems
CURRENT READING Blanchard, 5.2 & 5.3

Homework Assignments due Friday November 12
Section 5.1: 3, 4, 5, 18, 21.
Section 5.3: 2, 12, 13, 14, 17, 18.
Chapter 5 Review: 3, 4, 5, 6, 7, 8, 11, 12, 25, 27, 28.

SUMMARY
We shall continue our analysis of non-linear systems by introducing the concept of a Hamiltonian function.

Consider the following nonlinear planar system of ODEs

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= x - x^2\end{aligned}$$

Exercise

Show that the function $H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$ has the property that $\frac{dH}{dt} = 0$ if x and y simultaneously satisfy the given system of ODEs. (HINT: Use the Differentiation Chain Rule!)

1. The Hamiltonian

DEFINITION: Hamiltonian function

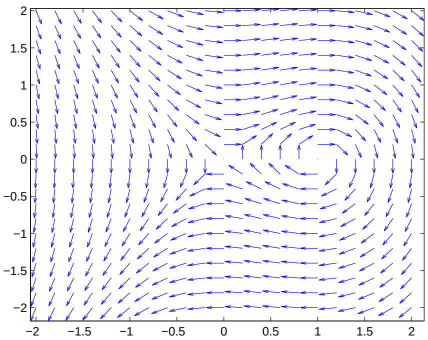
A real-valued function $H(x, y)$ is considered to be a conserved quantity for a system of ordinary differential equations if it is constant along ALL solution curves of the system. In other words, IF $(x(t), y(t))$ is a solution of the system then $H(x(t), y(t)$ is constant for all time which also implies that $\frac{d}{dt}H(x(t), y(t)) = 0$. The function $H(x, y)$ is known as the Hamiltonian function (or Hamiltonian) of the system of ODEs.

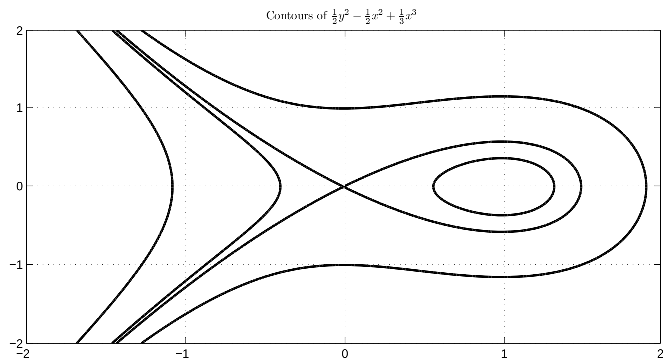
2. The Hamiltonian Level Curves and The Phase Portrait

RECALL

The **level curves** or **contours** of the function $H(x, y)$ are the set of points in the plane which atisfy the equation $H(x, y) = k$ for certain real values k.

Let's compare the level curves of $H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$ with the direction field of the system $\dot{x} = y; \quad \dot{y} = x - x^2$. What do you notice?





3. Hamiltonian System

DEFINITION: Hamiltonian System

A system ff differential equations is called a **Hamiltonian system** if there exists a real-valued function $H(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}$$

for all x and y. The function H is called the Hamiltonian function for the system.

EXAMPLE

The Hamiltonian often has a physical meaning for the sysem of ODEs that is modelling a partcular real-world situation, since it represents a quantity that is being conserved over time. For example, consider the system of ODEs that represents the **undamped** harmonic oscillator $y'' + qy = 0$:

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy$$

Let's show that the Hamiltonian for this system is $H(y, v) = \frac{1}{2}v^2 + \frac{q}{2}y^2$ which represents the total energy of the oscillator.

4. Obtaining Hamiltonians For Systems

In general the planar nonlinear system of first order DEs looks like

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

In order to find $H(x,y)$ we need to solve the following equations

$$f(x,y) = \frac{\partial H}{\partial y}$$
$$g(x,y) = -\frac{\partial H}{\partial x}$$

Does a Hamiltonian exist for this system? Well, if it does (and H has continuous second partial derivatives) then $\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$ which would mean that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} H_y = \frac{\partial}{\partial y} H_x = -\frac{\partial g}{\partial y}$$

So in order to check whether a given system of ODEs has a Hamiltonian or not all one needs to do is check whether

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$

Exercise

Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\frac{dx}{dt} = x + y^2$$
$$\frac{dy}{dt} = y^2 - x$$

EXAMPLE

Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\frac{dx}{dt} = -x\sin(y) + 2y$$
$$\frac{dy}{dt} = -\cos(y)$$

5. Equilibria of Hamiltonian Systems

Hamiltonian Systems Can Never Have Sources or Sinks As Equilibria. How could we prove that statement?

Consider

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial y}\\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x}\end{aligned}$$

at the point (x_0, y_0) which is the equilibrium point. Let's use the Linearization Technique!

The Jacobian of the linearized version of the Hamiltonian System at (x_0, y_0) will be

What can we say about its eigenvalues?

What does that allow us to conclude?