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# Differential Equations

Math 341 Fall 2010  
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MWF 2:30-3:25pm Fowler 307  
<http://faculty.oxy.edu/ron/math/341/10/>

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## Worksheet 18: Wednesday October 27

**TITLE** Linear Systems with Complex Eigenvalues

**CURRENT READING** Blanchard, 3.4

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### Homework Assignments due Friday October 29

Section 3.3: 3, 4, 7, 8.

Section 3.4: 1, 2, 3, 4.

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### SUMMARY

We'll continue to explore the various scenarios that occur with linear systems of ODEs. This time dealing with those that possess complex (or imaginary) eigenvalues.

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#### 1. Two Complex Eigenvalues

Given a system of linear ODEs with associated matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  the characteristic polynomial is  $p(\lambda) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - \text{tr}(A)\lambda + \det(A) = A\lambda^2 + B\lambda + C = 0$  where  $A = 1$ ,  $B = -(a + d)$  and  $C = ad - bc$ .

Using the quadratic formula to solve the characteristic quadratic polynomial equation  $A\lambda^2 + B\lambda + C = 0$  produces  $\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ .

Previously we showed that the conditions on  $A$ ,  $B$  and  $C$  for two real eigenvalues was  $B^2 > 4AC$  which is equivalent to  $(a - d)^2 > -4bc$ .

Clearly, the condition  $B^2 < 4AC$  which is equivalent to  $(a - d)^2 < -4bc$  will result in “imaginary” solutions to the characteristic polynomial, i.e. complex eigenvalues.

Also (for completeness), when  $(a - d)^2 = -4bc$  there will be only one solution to the quadratic equation, i.e. repeated eigenvalues equal to  $\lambda = \frac{(a + d)}{2}$ .

#### 2. Reviewing Complex Arithmetic

Recall that the  $i^2 = -1 \Leftrightarrow i = \sqrt{-1}$  and a complex number  $z$  has the form  $z = a + bi$  where  $a$  and  $b$  are real numbers. We refer to the real part of  $z$  as  $\text{Re}z$  and is equal to  $a$  and the complex part of  $z$  as  $\text{Im}z$  and is equal to  $b$ .

#### De Moivre's Formula

One of the most fun formulas in mathematics is DeMoivre's formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

When  $\theta = \pi$  one gets  $e^{i\pi} = -1$  which means that  $e^{i\pi} + 1 = 0$  which has all of these important numbers in it: 0, 1,  $\pi$ ,  $i$  and  $e$ . Some call it the greatest equation, ever!

**Complex Conjugates**

Complex numbers often show up in pairs, called complex conjugates. A complex conjugate of the number  $z = a + bi$  is  $z^* = a - bi$ . For example,  $\sqrt{-4} = 0 \pm 2i$ . These numbers  $-2i$  and  $2i$  are complex conjugates of each other. Note that the product of two complex conjugates is a completely real number.

**GROUPWORK**

Let's practice some complex arithmetic by simplifying some expressions into the form  $a + bi$ .

1.  $(1 + 2i)(1 - 2i)$

2.  $\frac{4}{i}$

3.  $e^{-i\pi/2}$

4.  $(-1 + 2i)(2 - 3i)$

**RECALL**

The general solution to  $\frac{d\vec{x}}{dt} = A\vec{x}$  is  $\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$  where  $A\vec{v}_1 = \lambda_1 \vec{v}_1$  and  $A\vec{v}_2 = \lambda_2 \vec{v}_2$ , i.e.  $\vec{v}_1$  and  $\vec{v}_2$  are eigenvectors corresponding to eigenvalues  $\lambda_1$  and  $\lambda_2$ .

What happens if the eigenvalues are complex? Let's find out!

**EXAMPLE**

What's the general solution to  $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \vec{x}$ . Check your answer.

### 3. Types Of Solutions With Complex Eigenvalues

Given that one has complex eigenvalues of the form  $\alpha \pm i\beta$  there are three possibilities for what the phase portrait behavior of the solution will look like. All the solutions will rotate around the origins with a period of  $\frac{2\pi}{\beta}$ .

**CASE 1:**  $\alpha < 0$  All solutions spiral inwards towards the origin, which is classified as a **stable spiral sink**.

**CASE 2:**  $\alpha > 0$  All solutions spiral outwards from the origin, which is classified as an **unstable spiral source**.

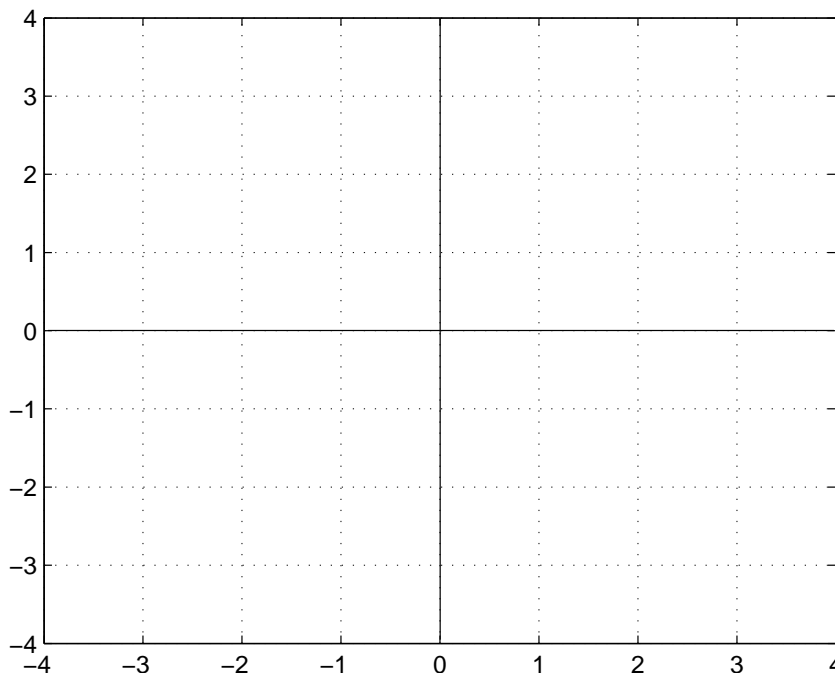
**CASE 3:**  $\alpha = 0$  All solutions follow elliptical or circular paths rotating around the origin, which is classified as an **unstable center**.

#### GROUPWORK

Come up with your own examples of 2x2 matrices  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  that have complex eigenvalues and the corresponding ODE for each of the three cases named above. Use `HPGSystemsSolver` or `ppplane` to help you sketch the phase portrait for each case on the given axes.

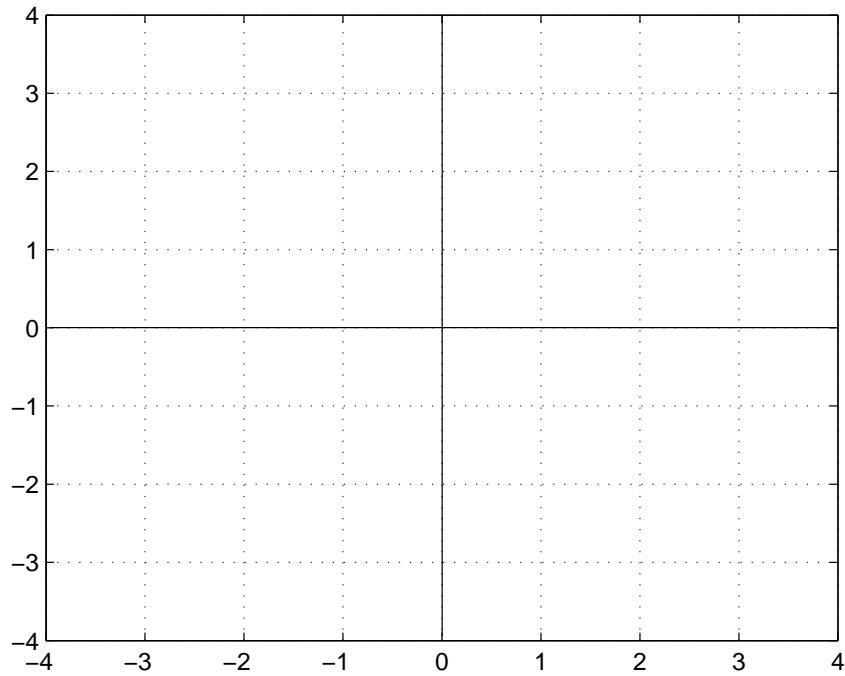
Recall,  $\lambda = \alpha \pm \beta i = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$ . What values of  $a, b, c$  and  $d$  will produce  $\alpha$  equalling zero?  $\alpha$  negative?  $\alpha$  positive? Or you could just explore with parameters until you obtain the phase portrait you are looking for in each case.

#### 4. CASE 1: spiral sink



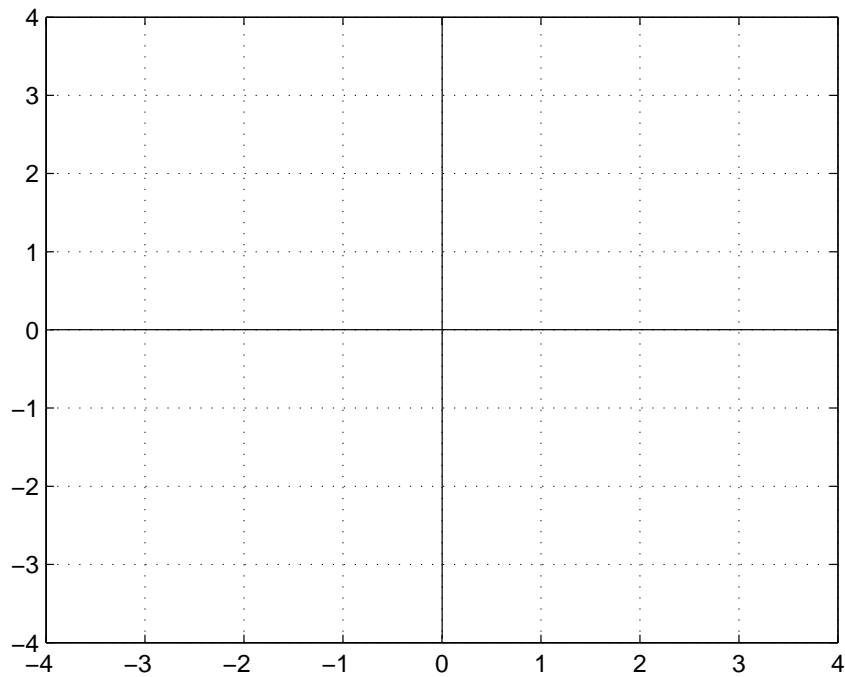
Corresponding ODE:  $\frac{d\vec{x}}{dt} = \begin{bmatrix} & \\ & \end{bmatrix} \vec{x}$

### 5. CASE 2: spiral source



Corresponding ODE:  $\frac{d\vec{x}}{dt} = \begin{bmatrix} & \\ & \end{bmatrix} \vec{x}$

### 6. CASE 3: unstable center



Corresponding ODE:  $\frac{d\vec{x}}{dt} = \begin{bmatrix} & \\ & \end{bmatrix} \vec{x}$