Checking the convolution property ($\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$) for Laplace transforms, we have

$$\mathcal{L}[f] = \frac{1}{s}, \quad \mathcal{L}[g] = \frac{1}{s+1},$$

and

$$\begin{aligned}\mathcal{L}[f * g] &= \frac{1}{s} - \frac{1}{s+1} \\ &= \frac{s+1-s}{s(s+1)} \\ &= \frac{1}{s(s+1)}.\end{aligned}$$

So, $\mathcal{L}[f] \cdot \mathcal{L}[g] = \mathcal{L}[f * g]$.

2. Using the definition of the convolution with f and g, we see that

$$\begin{aligned}(f * g)(t) &= \int_0^t e^{-a(t-u)} e^{-bu}\, du \\ &= \int_0^t e^{-at} e^{(a-b)u}\, du \\ &= -e^{-at} \left. \frac{e^{(a-b)u}}{a-b} \right|_0^t \\ &= e^{-at} \left(\frac{e^{(a-b)t}}{a-b} - \frac{1}{a-b} \right) \\ &= \frac{e^{-bt}}{a-b} - \frac{e^{-at}}{a-b}.\end{aligned}$$

Checking the convolution property ($\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$) for Laplace transforms, we have

$$\mathcal{L}[f] = \frac{1}{s+a}, \quad \mathcal{L}[g] = \frac{1}{s+b},$$

and

$$\begin{aligned}\mathcal{L}[f * g] &= \frac{1}{(s+b)(a-b)} - \frac{1}{(s+a)(a-b)} \\ &= \frac{s+a}{(s+a)(s+b)(a-b)} - \frac{s+b}{(s+a)(s+b)(a-b)} \\ &= \frac{a-b}{(s+a)(s+b)(a-b)} \\ &= \frac{1}{(s+a)(s+b)}.\end{aligned}$$

Therefore, $\mathcal{L}[f] \cdot \mathcal{L}[g] = \mathcal{L}[f * g]$.

So

$$\int_0^t u_2(t-v)u_3(v)\,dv = \begin{cases} 0, & \text{if } t < 5, \\ \displaystyle\int_3^{t-2} 1\,dv, & \text{if } t \geq 5. \end{cases}$$

Evaluating the second integral, we get

$$\int_3^{t-2} 1\,dv = t\Big|_3^{t-2} = t - 5.$$

We have a function that is 0 for $t < 5$ and equal to $t - 5$ for $t \geq 5$, so our function is $u_5(t)(t-5)$.

Checking the convolution property ($\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$) for Laplace transforms, we have

$$\mathcal{L}[f] = \frac{e^{-2s}}{s}, \quad \mathcal{L}[g] = \frac{e^{-3s}}{s},$$

and

$$\mathcal{L}[f * g] = \frac{e^{-5s}}{s^2}.$$

So, $\mathcal{L}[f] \cdot \mathcal{L}[g] = \mathcal{L}[f * g]$.

5. Using the definition of the convolution with f and g, we see that

$$(f * g)(t) = \int_0^t 3\sin(t-u)\cos(2u)\,du.$$

We will use four trigonometric identities to evaluate this integral:

$$\sin(t-u) = \sin t \cos u - \cos t \sin u$$

$$\sin(mt)\sin(nt) = \tfrac{1}{2}\left[\cos((m-n)t) - \cos((m+n)t)\right]$$

$$\cos(mt)\cos(nt) = \tfrac{1}{2}\left[\cos((m+n)t) + \cos((m-n)t)\right]$$

$$\sin(mt)\cos(nt) = \tfrac{1}{2}\left[\sin((m+n)t) + \sin((m-n)t)\right].$$

So

$$\int_0^t 3\sin(t-u)\cos(2u)\,du$$

$$= \int_0^t \left[3\cos 2u \cos u \sin t - \cos 2u \sin u \cos t\right] du$$

$$= \int_0^t \left[\tfrac{3}{2}(\cos 3u + \cos u)\sin t - \tfrac{3}{2}(\sin 3u - \sin u)\cos t\right] du$$

$$= \sin t \left[\tfrac{1}{2}\sin 3u + \tfrac{3}{2}\sin u\right]_0^t + \cos t \left[\tfrac{1}{2}\cos 3u - \tfrac{3}{2}\cos u\right]_0^t$$

$$= \sin t \left(\tfrac{1}{2}\sin 3t + \tfrac{3}{2}\sin t\right) + \cos t \left(\tfrac{1}{2}\cos 3t - \tfrac{3}{2}\cos t + 1\right)$$

$$= \tfrac{1}{2}\sin 3t\sin t + \tfrac{3}{2}\sin^2 t + \tfrac{1}{2}\cos 3t\cos t - \tfrac{3}{2}\cos^2 t + \cos t$$

$$= \tfrac{1}{4}(\cos 2t - \cos 4t) + \tfrac{3}{2}\sin^2 t + \tfrac{1}{4}(\cos 4t + \cos 2t) - \tfrac{3}{2}\cos^2 t + \cos t$$

$$= \tfrac{1}{2}\cos 2t + \tfrac{3}{2}\left(\sin^2 t - \cos^2 t\right) + \cos t$$

$$= \tfrac{1}{2}\cos 2t - \tfrac{3}{2}\cos 2t + \cos t$$

$$= \cos t - \cos 2t,$$

which is the same answer obtained in the text using the technique of Laplace transforms.

6. We will use the substitution $v = t - u$, so that $u = t - v$, and $du = -dv$. Also, as u goes from 0 to t, v goes from t to 0, so we have

$$\begin{aligned}(f * g)(t) &= \int_0^t f(t-u)g(u)\,du \\ &= -\int_t^0 f(v)g(t-v)\,dv \\ &= \int_0^t f(v)g(t-v)\,dv \\ &= (g * f)(t).\end{aligned}$$

7. Taking Laplace transform of both sides of the equation and solving for $\mathcal{L}[\zeta]$ (see page 607), we obtain

$$\mathcal{L}[\zeta] = \frac{1}{s^2 + ps + q}.$$

Hence, if we let

$$z(s) = s^2 + ps + q,$$

we have that $z(0) = 5$ and $z(2) = 17$. Now $z(0) = 5$ implies $q = 5$. Using $z(2) = 17 = 2^2 + 2p + 5$, we see that $p = 4$.

8. Since $\eta(t)$ solves the first equation, we know that

$$\frac{d\eta}{dt} + a\eta = f(t), \quad \eta(0) = 0.$$

Taking the Laplace transform of both sides of the equation, we get

$$s\mathcal{L}[\eta] - \eta(0) + a\mathcal{L}[\eta] = \mathcal{L}[f].$$

Substituting the initial condition and solving for $\mathcal{L}[\eta]$, we have

$$\mathcal{L}[\eta] = \frac{\mathcal{L}[f]}{s + a}.$$

Now, since $\zeta(t)$ solves the second equation, we know that

$$\frac{d\zeta}{dt} + a\zeta = \delta_0.$$

So

$$s\mathcal{L}[\zeta] + a\mathcal{L}[\zeta] = \mathcal{L}[\delta_0],$$

and

$$\mathcal{L}[\zeta] = \frac{1}{s+a}.$$

Hence,

$$\mathcal{L}[\zeta]\cdot\mathcal{L}[f] = \mathcal{L}[\eta].$$

9. **(a)** Since ζ solves the initial-value problem above, we know that

$$\frac{d\zeta^2}{dt^2} + p\frac{d\zeta}{dt} + q\zeta = \delta_0(t), \quad \zeta(0) = \zeta'(0) = 0^-.$$

Taking Laplace transforms of both sides, and substituting initial conditions gives us

$$s^2\mathcal{L}[\zeta] + ps\mathcal{L}[\zeta] + q\mathcal{L}[\zeta] = 1,$$

which yields

$$\mathcal{L}[\zeta] = \frac{1}{s^2 + ps + q}.$$

Now, taking Laplace transforms of both sides of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0, \quad y(0) = a,\ y'(0) = 0$$

gives us

$$s^2\mathcal{L}[y] - sa + ps\mathcal{L}[y] - pa + q\mathcal{L}[y] = 0.$$

Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{a(s+p)}{s^2 + ps + q},$$

so

$$\mathcal{L}[y] = a(s+p)\mathcal{L}[\zeta].$$

(b) Taking Laplace transforms of both sides of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0, \quad y(0) = 0,\ y'(0) = b$$

gives us

$$s^2\mathcal{L}[y] - b + ps\mathcal{L}[y] + q\mathcal{L}[y] = 0.$$

Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{b}{s^2 + ps + q},$$

so

$$\mathcal{L}[y] = b\mathcal{L}[\zeta].$$

(c) Taking Laplace transforms of both sides of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t), \quad y(0) = a, y'(0) = b$$

gives us

$$s^2\mathcal{L}[y] - sa - b + ps\mathcal{L}[y] - pa + q\mathcal{L}[y] = \mathcal{L}[f].$$

Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{\mathcal{L}[f] + a(s+p) + b}{s^2 + ps + q},$$

so

$$\mathcal{L}[y] = (\mathcal{L}[f] + a(s+p) + b)\,\mathcal{L}[\zeta].$$

10. Since η solves the first initial-value problem, we know that

$$\frac{d\eta^2}{dt^2} + p\frac{d\eta}{dt} + q\eta = u_0(t), \quad \eta(0) = \eta'(0) = 0^-.$$

Taking Laplace transforms of both sides and replacing the initial conditions gives us

$$s^2\mathcal{L}[\eta] + ps\mathcal{L}[\eta] + q\mathcal{L}[\eta] = \frac{1}{s}.$$

Solving for $\mathcal{L}[\eta]$ gives

$$\mathcal{L}[\eta] = \frac{1}{s(s^2 + ps + q)}.$$

If we take the Laplace transform of both sides of the second initial-value problem, and solve for $\mathcal{L}[y]$, we have

$$\mathcal{L}[y] = \frac{\mathcal{L}[f]}{s^2 + ps + q}.$$

Using the convolution property for Laplace Transforms, we get

$$\begin{aligned}\mathcal{L}[y] &= s\,(\mathcal{L}[f] \cdot \mathcal{L}[\eta]) \\ &= s\,(\mathcal{L}[f * \eta])\,.\end{aligned}$$

Now,

$$(f * \eta)(t) = \int_0^t f(t-u)\eta(u)\,du,$$

so

$$(f * \eta)(0) = \int_0^0 f(t-u)\eta(u)\,du = 0.$$

Using the rule that

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0),$$

we have that

$$\begin{aligned}\mathcal{L}\left[\frac{d}{dt}(f * \eta)\right] &= s\mathcal{L}[(f * \eta)] - (f * \eta)(0) \\ &= s\mathcal{L}[(f * \eta)] \\ &= \mathcal{L}[y].\end{aligned}$$