2. Since $y(0) = 1$ is between the equilibrium solutions $y_2(t) = 0$ and $y_3(t) = 2$, we must have $0 < y(t) < 2$ for all t because the Uniqueness Theorem implies that graphs of solutions cannot cross (or even touch in this case).

3. Because $y_2(0) < y(0) < y_1(0)$, we know that

$$-t^2 = y_2(t) < y(t) < y_1(t) = t + 2$$

for all t. This restricts how large positive or negative $y(t)$ can be for a given value of t (that is, between $-t^2$ and $t+2$). As $t \to -\infty$, $y(t) \to -\infty$ between $-t^2$ and $t+2$ ($y(t) \to -\infty$ as $t \to -\infty$ at least linearly, but no faster than quadratically).

4. Because $y_1(0) < y(0) < y_2(0)$, the solution $y(t)$ must satisfy $y_1(t) < y(t) < y_2(t)$ for all t by the Uniqueness Theorem. Hence $-1 < y(t) < 1 + t^2$ for all t.

5. The Existence Theorem implies that a solution with this initial condition exists, at least for a small t-interval about $t = 0$. This differential equation has equilibrium solutions $y_1(t) = 0$, $y_2(t) = 2$, and $y_3(t) = 3$. Since $y(0) = 4$, the Uniqueness Theorem implies that $y(t) > 3$ for all t in the domain of $y(t)$. Also, $dy/dt > 0$ for all $y > 3$, so the solution $y(t)$ is increasing for all t in its domain.

6. Note that $dy/dt = 0$ if $y = 3$. Hence, $y_1(t) = 3$ for all t is an equilibrium solution. By the Uniqueness Theorem, this is the only solution that is 3 at $t = 0$. Therefore, $y(t) = 3$ for all t.

7. Because $0 < y(0) < 2$ and $y_1(t) = 0$ and $y_2(t) = 2$ are equilibrium solutions of the differential equation, we know that $0 < y(t) < 2$ for all t by the Uniqueness Theorem. Also, $dy/dt > 0$ for $0 < y < 2$, so dy/dt is always positive for this solution. Hence, $y(t) \to 2$ as $t \to \infty$, and $y(t) \to 0$ as $t \to -\infty$.

8. Note that $y(0) < 0$. Since $y_1(t) = 0$ is an equilibrium solution, the Uniqueness Theorem implies that $y(t) < 0$ for all t. Also, $dy/dt < 0$ if $y < 0$, so $y(t)$ is decreasing for all t, and $y(t) \to -\infty$ as t increases. As $t \to -\infty$, $y(t) \to 0$.

9. **(a)** To check that $y_1(t) = t^2$ is a solution, we compute

$$\frac{dy_1}{dt} = 2t$$

and

$$\begin{aligned}-y_1^2 + y_1 + 2y_1t^2 + 2t - t^2 - t^4 &= -(t^2)^2 + (t^2) + 2(t^2)t^2 + 2t - t^2 - t^4 \\ &= 2t.\end{aligned}$$

To check that $y_2(t) = t^2 + 1$ is a solution, we compute

$$\frac{dy_2}{dt} = 2t$$

and

$$\begin{aligned}-y_2^2 + y_2 + 2y_2t^2 + 2t - t^2 - t^4 &= -(t^2+1)^2 + (t^2+1) + 2(t^2+1)t^2 \\ &\quad + 2t - t^2 - t^4 \\ &= 2t.\end{aligned}$$

12. **(a)** Note that

$$\frac{dy_1}{dt} = \frac{d}{dt}\left(\frac{1}{t-1}\right) = -\frac{1}{(t-1)^2} = -(y_1(t))^2$$

and

$$\frac{dy_2}{dt} = \frac{d}{dt}\left(\frac{1}{t-2}\right) = -\frac{1}{(t-2)^2} = -(y_2(t))^2,$$

so both $y_1(t)$ and $y_2(t)$ are solutions.

(b) Note that $y_1(0) = -1$ and $y_2(0) = -1/2$. If $y(t)$ is another solution whose initial condition satisfies $-1 < y(0) < -1/2$, then $y_1(t) < y(t) < y_2(t)$ for all t by the Uniqueness Theorem. Also, since $dy/dt < 0$, $y(t)$ is decreasing for all t in its domain. Therefore, $y(t) \to 0$ as $t \to -\infty$, and the graph of $y(t)$ has a vertical asymptote between $t = 1$ and $t = 2$.

13. The key observation is that the differential equation is not defined when $t = 0$.

(a) Note that $dy_1/dt = 0$ and $y_1/t^2 = 0$, so $y_1(t)$ is a solution.

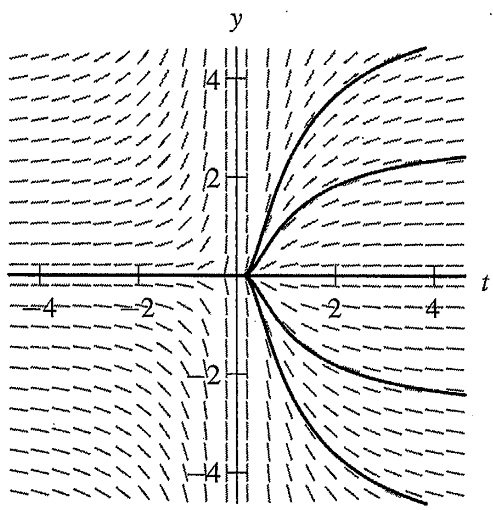
(b) Separating variables, we have

$$\int \frac{dy}{y} = \int \frac{dt}{t^2}.$$

Solving for y we obtain $y(t) = ce^{-1/t}$, where c is any constant. Thus, for any real number c, define the function $y_c(t)$ by

$$y_c(t) = \begin{cases} 0 & \text{for } t \leq 0; \\ ce^{-1/t} & \text{for } t > 0. \end{cases}$$

For each c, $y_c(t)$ satisfies the differential equation for all $t \neq 0$.



There are infinitely many solutions of the form $y_c(t)$ that agree with $y_1(t)$ for $t < 0$.

(c) Note that $f(t, y) = y/t^2$ is not defined at $t = 0$. Therefore, we *cannot* apply the Uniqueness Theorem for the initial condition $y(0) = 0$. The "solution" $y_c(t)$ given in part (b) actually represents two solutions, one for $t < 0$ and one for $t > 0$.

14. **(a)** The equation is separable. We separate the variables and compute

$$\int y^{-3}\,dy = \int dt.$$

Solving for y, we obtain

$$y(t) = \frac{1}{\sqrt{c - 2t}}$$

for any constant c. To find the desired solution, we use the initial condition $y(0) = 1$ and obtain $c = 1$. So the solution to the initial-value problem is

$$y(t) = \frac{1}{\sqrt{1 - 2t}}.$$

(b) This solution is defined when $-2t + 1 > 0$, which is equivalent to $t < 1/2$.

(c) As $t \to 1/2^-$, the denominator of $y(t)$ becomes a small positive number, so $y(t) \to \infty$. We only consider $t \to 1/2^-$ because the solution is defined only for $t < 1/2$. (The other "branch" of the function is also a solution, but the solution that includes $t = 0$ in its domain is not defined for $t \geq 1/2$.) As $t \to -\infty$, $y(t) \to 0$.

15. **(a)** The equation is separable, so we obtain

$$\int (y + 1)\,dy = \int \frac{dt}{t - 2}.$$

Solving for y with help from the quadratic formula yields the general solution

$$y(t) = -1 \pm \sqrt{1 + \ln(c(t - 2)^2)}$$

where c is a constant. Substituting the initial condition $y(0) = 0$ and solving for c, we have

$$0 = -1 \pm \sqrt{1 + \ln(4c)},$$

and thus $c = 1/4$. The desired solution is therefore

$$y(t) = -1 + \sqrt{1 + \ln((1 - t/2)^2)}$$

(b) The solution is defined only when $1 + \ln((1 - t/2)^2) \geq 0$, that is, when $|t - 2| \geq 2/\sqrt{e}$. Therefore, the domain of the solution is

$$t \leq 2(1 - 1/\sqrt{e}).$$

(c) As $t \to 2(1 - 1/\sqrt{e})$, then $1 + \ln((1 - t/2)^2) \to 0$. Thus

$$\lim_{t \to 2(1-1/\sqrt{e})} y(t) = -1.$$

Note that the differential equation is not defined at $y = -1$. Also, note that

$$\lim_{t \to -\infty} y(t) = \infty.$$