REVIEW EXERCISES FOR CHAPTER 3

1. The characteristic polynomial is $(1-\lambda)(2-\lambda)$, so the eigenvalues are $\lambda = 1$ and $\lambda = 2$.

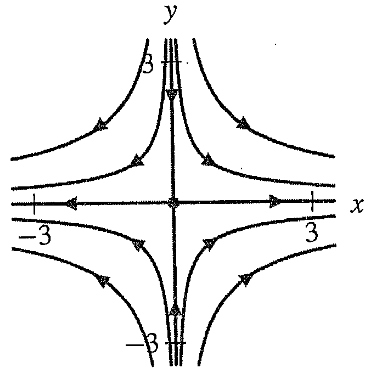
2. The characteristic polynomial is

$$(-\lambda)(-\lambda) - (1)(2) = \lambda^2 - 2,$$

so the eigenvalues are $\lambda = \pm\sqrt{2}$.

3. The system has eigenvalues -2 and 3. One eigenvector associated with $\lambda = 3$ is $(1, 0)$, and one eigenvector associated with $\lambda = -2$ is $(0, 1)$. The general solution is

$$\mathbf{Y}(t) = k_1 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



4. By definition, the zero vector, $\mathbf{Y}_1$, is never an eigenvector. We can check the others by computing $\mathbf{AY}$. For example,

$$\mathbf{AY}_2 = \mathbf{A}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \mathbf{Y}_2,$$

so $\mathbf{Y}_2$ is an eigenvector (with eigenvalue $\lambda = 1$). On the other hand,

$$\mathbf{AY}_3 = \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

which is not a scalar multiple of $\mathbf{Y}_3$, so $\mathbf{Y}_3$ is not an eigenvector. Also, $\mathbf{AY}_4 = 3\mathbf{Y}_4$, so $\mathbf{Y}_4$ is an eigenvector (with eigenvalue $\lambda = 3$). Since we know that $\mathbf{Y}_2$ is an eigenvector and $\mathbf{Y}_5 = -2\mathbf{Y}_2$, $\mathbf{Y}_5$ is also an eigenvector. The vectors $\mathbf{Y}_2$ and $\mathbf{Y}_4$ are two linearly independent eigenvectors corresponding to different eigenvalues. Therefore, $\mathbf{Y}_6$ cannot be an eigenvector because it is neither a scalar multiple of $\mathbf{Y}_2$ nor $\mathbf{Y}_4$.

5. Note that $b \geq 0$ by assumption. The characteristic polynomial is

$$s^2 + bs + 5,$$

so the eigenvalues are

$$s = \frac{-b \pm \sqrt{b^2 - 20}}{2}.$$

If $b > \sqrt{20}$, the harmonic oscillator is overdamped. If $b = \sqrt{20}$, the harmonic oscillator is critically damped. If $0 < b < \sqrt{20}$, the harmonic oscillator is underdamped, and if $b = 0$, the harmonic oscillator is undamped.

6. Written in coordinates, the system is $dx/dt = 0$ and $dy/dt = x - y$. Hence, the equilibrium points are all points on the line $y = x$.

7. Every linear system has the origin as an equilibrium point, so the solution to the initial-value problem is the equilibrium solution $\mathbf{Y}(t) = (0, 0)$ for all t.

8. If $k > 0$, the general solution is

$$y(t) = c_1 \sin \sqrt{k}\, t + c_2 \cos \sqrt{k}\, t.$$

If $k < 0$, the general solution is

$$y(t) = c_1 e^{\sqrt{-k}\, t} + c_2 e^{-\sqrt{-k}\, t}.$$

The general solution for $k = 0$ is $y(t) = c_1 t + c_2$. Hence, only (b) and (d) are solutions under the given assumptions on k.

9. Letting $x(t) = 2\cos t$ and $y(t) = \sin t$, we have

$$\frac{dx}{dt} = \frac{d(2\cos t)}{dt} = -2\sin t = -2y$$

and

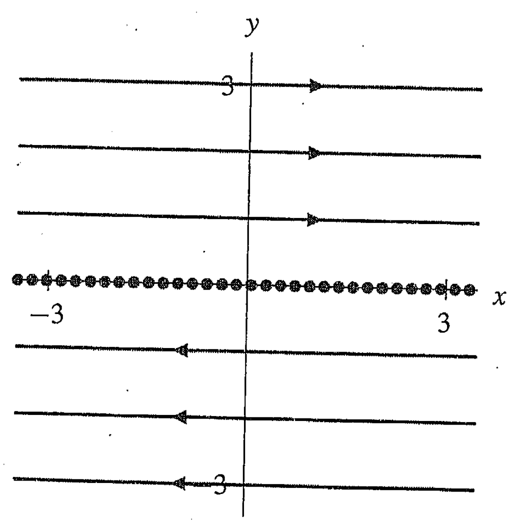
$$\frac{dy}{dt} = \frac{d(\sin t)}{dt} = \cos t = \frac{x}{2}.$$

Hence, $\mathbf{Y}(t)$ satisfies the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & -2 \\ 1/2 & 0 \end{pmatrix} \mathbf{Y}.$$

10. Written in terms of coordinates, the system is $dx/dt = y$ and $dy/dt = 0$. From the second equation, we see that $y(t) = k_2$, where k_2 is an arbitrary constant. Then $x(t) = k_2 t + k_1$, where k_1 is another arbitrary constant. In vector notation, the general solution is

$$\mathbf{Y}(t) = \begin{pmatrix} k_2 t + k_1 \\ k_2 \end{pmatrix}.$$



11. False. For example, the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{Y}$$

has a line of equilibria (the y-axis). Another example is the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{Y}.$$

Every point is an equilibrium point for this system.

12. True. If $\mathbf{A}$ is the matrix and λ is the eigenvalue associated to $\mathbf{Y}_0$, then

$$\mathbf{A}(k\mathbf{Y}_0) = k\mathbf{A}\mathbf{Y}_0 = k\lambda\mathbf{Y}_0 = \lambda(k\mathbf{Y}_0).$$

Consequently, $k\mathbf{Y}_0$ is an eigenvector as long as $k \neq 0$. (Note that $k = 0$ is excluded because the zero vector is never an eigenvector by definition.)

13. True. Linear systems have solutions that consist of just sine and cosine functions only when the eigenvalues are purely imaginary (that is, of the form $\pm i\omega$). In this case, the sine and cosine terms are of the form $\sin \omega t$ and $\cos \omega t$. For the first coordinate of $\mathbf{Y}(t)$ to be part of a solution, we would have to have $\omega = 2$, but the second coordinate would force $\omega = 1$. So this function cannot be the solution of a linear system.

14. False. The graph has $y(0) = 0$ and $y'(0) = 0$. However, these values are the initial conditions for the equilibrium solution $y(t) = 0$ for all t.

15. False. In the graph, the amount of time between consecutive crossings of the t-axis decreases as t increases. Even though solutions of underdamped harmonic oscillators oscillate, the amount of time between consecutive crossings of the t-axis is constant.

16. True. The period of solutions is $2\pi/\sqrt{k}$, so if k increases, then the period decreases. Consequently, the time between successive maxima decreases.

17. True. If the matrix has a real eigenvalue, then there the corresponding system has at least a line of eigenvectors. Any initial condition that is an eigenvector corresponds to a solution that stays on the line of eigenvectors for all time. Hence, a system with a real eigenvalue has infinitely many straight-line solutions.

18. True. The functions that arise in solutions of a linear system are linear combinations of $e^{\lambda t}$, $\cos \beta t$, $\sin \beta t$, and $te^{\lambda t}$. All of these functions are defined for all t. Consequently, all solutions of $d\mathbf{Y}/dt = \mathbf{AY}$ are defined for all t.

19. First, we compute the characteristic polynomials and eigenvalues for each matrix.

(i) The characteristic polynomial is $\lambda^2 + 1$, and the eigenvalues are $\lambda = \pm i$. Center.

(ii) The characteristic polynomial is $\lambda^2 + 2\lambda - 2$, and the eigenvalues are $\lambda = 1 \pm \sqrt{3}$. Saddle.

(iii) The characteristic polynomial is $\lambda^2 + 3\lambda + 1$, and the eigenvalues are $\lambda = (-3 \pm \sqrt{5})/2$. Sink.

(iv) The characteristic polynomial is $\lambda^2 + 1$, and the eigenvalues are $\lambda = \pm i$. Center.

(v) The characteristic polynomial is $\lambda^2 - \lambda - 2$, and the eigenvalues are $\lambda = -1$ and $\lambda = 2$. Saddle.

(vi) The characteristic polynomial is $\lambda^2 - 3\lambda + 1$, and the eigenvalues are $\lambda = (3 \pm \sqrt{5})/2$. Source.

(vii) The characteristic polynomial is $\lambda^2 + 4\lambda + 4$. The eigenvalue $\lambda = -2$ is a repeated eigenvalue. Sink.

(viii) The characteristic polynomial is $\lambda^2 + 2\lambda + 3$, and the eigenvalues are $\lambda = -1 \pm i\sqrt{2}$. Spiral sink.

Given this information, we can match the matrices with the phase portraits.

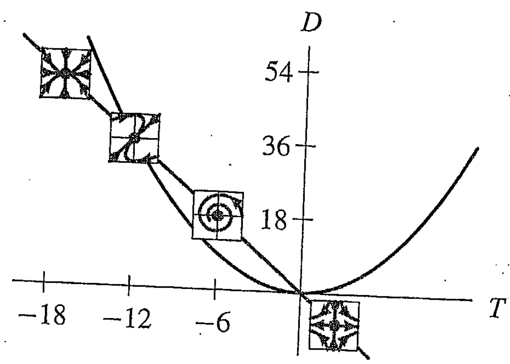
(a) This portrait is a center. There are two possibilities, (i) and (iv). At $(1, 0)$, the vector for (i) is $(1, -2)$, and the vector for (iv) is $(-1, -2)$. This phase portrait corresponds to matrix (iv).

(b) This portrait is a sink with two lines of eigenvectors. The only possibility is matrix (iii).

(c) This portrait is a saddle. The only possibilities are (ii) and (v). However, in (v), all vectors on the y-axis are eigenvectors corresponding to the eigenvalue $\lambda = -1$. Therefore, the phase portrait cannot correspond to (v).

(d) This portrait is a sink with a single line of eigenvectors. The only possibility is matrix (vii).

20. **(a)** The trace T is a, and the determinant D is $-3a$. Therefore, the curve in the trace-determinant plane is $D = -3T$.



(b) The line $D = -3T$ crosses the parabola $T^2 - 4D = 0$ at two points—at $(T, D) = (-12, 36)$ if $a = -12$ and at $(T, D) = (0, 0)$ if $a = 0$. Therefore, bifucations occur at $a = -12$ and at $a = 0$. The portion of the line for which $a < -12$ corresponds to a positive determinant and a negative trace such that $T^2 - 4D < 0$. The corresponding phase portraits are real sinks. If $a = -12$, we have a sink with repeated eigenvalues. If $-12 < a < 0$, we have complex eigenvalues with negative real parts. Therefore, the phase portraits are spiral sinks. If $a = 0$, we have a degenerate case where the y-axis is an entire line of equilibrium points. Finally, if $a > 0$, the corresponding portion of the line is below the T-axis, and the phase portraits are saddles.

21. First, we compute the characteristic polynomials and eigenvalues for each matrix.

(i) The characteristic polynomial is $\lambda^2 - 3\lambda - 4$, and the eigenvalues are $\lambda = -1$ and $\lambda = 4$. Saddle.

(ii) The characteristic polynomial is $\lambda^2 - 7\lambda + 10$, and the eigenvalues are $\lambda = 2$ and $\lambda = 5$. Source.

(iii) The characteristic polynomial is $\lambda^2 + 4\lambda + 3$, and the eigenvalues are $\lambda = -3$ and $\lambda = -1$. Sink.

(iv) The characteristic polynomial is $\lambda^2 + 4$, and the eigenvalues are $\lambda = \pm 2i$. Center.

(v) The characteristic polynomial is $\lambda^2 + 9$, and the eigenvalues are $\lambda = \pm 3i$. Center.

(vi) The characteristic polynomial is $\lambda^2 - 2\lambda + \frac{15}{16}$, and the eigenvalues are $\lambda = 3/4$ and $\lambda = 5/4$. Source.

REVIEW EXERCISES FOR CHAPTER 4

1. The natural guess for a particular solution is a constant function $y_p(t) = a$. For this function, $dy_p/dt = d^2y_p/dt^2 = 0$, so it is a solution if and only if $ka = 1$. Hence, the constant function $y_p(t) = 1/k$ for all t is a solution.

2. The angular frequency ω of the forcing function is the same as the natural frequency of the oscillator if $\omega^2 = 4$. Hence the oscillator is in resonance if $\omega = \pm 2$. (Note that $\cos(-2t) = \cos(2t)$, so including the $\pm$ sign is optional.)

3. The frequency of the steady-state solution is the same as the frequency of the forcing function. The frequency of $4\cos 2t$ is $2/(2\pi) = 1/\pi$.

4. The system corresponding to this equation is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -4y + \sin t,$$

and there are no points (y, v) for which the right-hand sides of both equations are zero for all t. Hence there are no equilibrium solutions of this equation.

5. Yes, there is a steady-state response for this equation. Note that the general solution to the associated homogeneous equation is $ke^{-\lambda t}$, which tends to zero as $t \to \infty$.

We can compute one particular solution to forced equation using the techniques of Section 1.8, or we can complexify the equation and guess a solution to

$$\frac{dy}{dt} + \lambda y = e^{i\omega t}.$$

To find one solution to the complexified equation, we guess $y_c(t) = ae^{i\omega t}$. Then

$$\frac{dy_c}{dt} + \lambda y_c = a(i\omega)e^{i\omega t} + \lambda(ae^{i\omega t})$$
$$= a(\lambda + i\omega)e^{i\omega t}.$$

We have a solution if $a = 1/(\lambda + i\omega)$.

To obtain the steady-state solution of the original equation, we take the real part of

$$y_c(t) = \frac{1}{\lambda + i\omega} e^{i\omega t} = \frac{\lambda - i\omega}{\lambda^2 + \omega^2} e^{i\omega t}.$$

We obtain

$$\frac{\lambda}{\lambda^2 + \omega^2} \cos \omega t + \frac{\omega}{\lambda^2 + \omega^2} \sin \omega t.$$

See the discussion and example on page 120 of Section 1.8.

6. No. Resonance occurs if the frequency of the forcing matches the natural frequency of the equation. Solutions of

$$\frac{dy}{dt} + \lambda y = 0$$