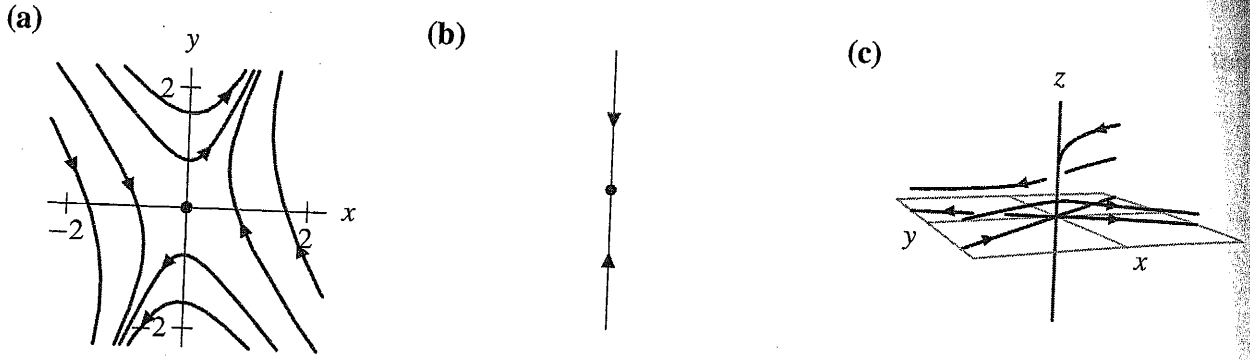
5.



REVIEW EXERCISES FOR CHAPTER 2

1. The simplest solution is an equilibrium solution, and the origin is an equilibrium point for this system. Hence, the equilibrium solution $(x(t), y(t)) = (0, 0)$ for all t is a solution.

2. Note that $dy/dt > 0$ for all (x, y). Hence, there are no equilibrium points for this system.

3. Let $v = dy/dt$. Then $dv/dt = d^2y/dt^2$, and we obtain the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= 1. \end{aligned}$$

4. First we solve $dv/dt = 1$ and get $v(t) = t + c_1$, where c_1 is an arbitrary constant. Next we solve $dy/dt = v = t + c_1$ and obtain $y(t) = \frac{1}{2}t^2 + c_1 t + c_2$, where c_2 is an arbitrary constant. Therefore, The general solution of the system is

$$\begin{aligned} y(t) &= \tfrac{1}{2}t^2 + c_1 t + c_2 \\ v(t) &= t + c_1. \end{aligned}$$

5. The equation for dx/dt gives $y = 0$. If $y = 0$, then $\sin(xy) = 0$, so $dy/dt = 0$. Hence, every point on the x-axis is an equilibrium point.

6. The second-order equation for this harmonic oscillator is

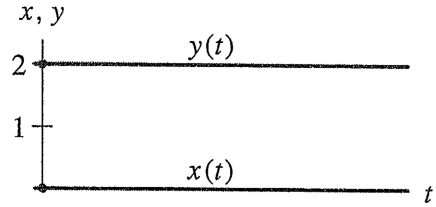
$$\beta\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \alpha y = 0.$$

The corresponding system is

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{\alpha}{\beta}y - \frac{\gamma}{\beta}v. \end{aligned}$$

7. From the equation for dx/dt, we know that $x(t) = k_1 e^{2t}$, where k_1 is an arbitrary constant, and from the equation for dy/dt, we have $y(t) = k_2 e^{-3t}$, where k_2 is another arbitrary constant. The general solution is $(x(t), y(t)) = (k_1 e^{2t}, k_2 e^{-3t})$.

8. Note that (0, 2) is an equilibrium point for this system. Hence, the solution with this initial condition is an equilibrium solution.



9. There are many examples. One is

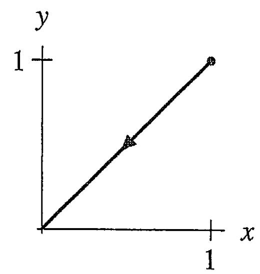
$$\frac{dx}{dt} = (x^2 - 1)(x^2 - 4)(x^2 - 9)(x^2 - 16)(x^2 - 25)$$
$$\frac{dy}{dt} = y.$$

This system has equilibria at $(\pm 1, 0)$, $(\pm 2, 0)$, $(\pm 3, 0)$, $(\pm 4, 0)$, and $(\pm 5, 0)$.

10. One step of Euler's method is

$$\begin{aligned}(2, 1) + \Delta t\, \mathbf{F}(2, 1) &= (2, 1) + 0.5\,(3, 2) \\ &= (3.5, 2).\end{aligned}$$

11. The point (1, 1) is on the line $y = x$. Along this line, the vector field for the system points toward the origin. Therefore, the solution curve consists of the half-line $y = x$ in the first quadrant. Note that the point (0, 0) is not on this curve.



12. Let $\mathbf{F}(x, y) = (f(x, y), g(x, y))$ be the vector field for the original system. The vector field for the new system is

$$\begin{aligned}\mathbf{G}(x, y) &= (-f(x, y), -g(x, y)) \\ &= -(f(x, y), g(x, y)) \\ &= -\mathbf{F}(x, y).\end{aligned}$$

In other words, the directions of vectors in the new field are the opposite of the directions in the original field. Consequently, the phase portrait of new system has the same solution curves as the original phase portrait except that their directions are reversed. Hence, all solutions tend away from the origin as t increases.

13. True. First, we check the equation for dx/dt. We have

$$\frac{dx}{dt} = \frac{d(e^{-6t})}{dt} = -6e^{-6t},$$

and

$$2x - 2y^2 = 2(e^{-6t}) - 2(2e^{-3t})^2 = 2e^{-6t} - 8e^{-6t} = -6e^{-6t}.$$

Since that equation holds, we check the equation for dy/dt. We have

$$\frac{dy}{dt} = \frac{d(2e^{-3t})}{dt} = -6e^{-3t},$$

and

$$-3y = -3(2e^{-3t}) = -6e^{-3t}.$$

Since the equations for both dx/dt and dy/dt hold, the function $(x(t), y(t)) = (e^{-6t}, 2e^{-3t})$ is a solution of this system.

14. False. A solution to this system must consist of a pair $(x(t), y(t))$ of functions.

15. False. The components of the vector field are the right-hand sides of the equations of the system.

16. True. For example,

$$\frac{dx}{dt} = y \qquad \text{and} \qquad \frac{dx}{dt} = 2y$$
$$\frac{dy}{dt} = x \qquad \qquad\qquad \frac{dy}{dt} = 2x$$

have the same direction field. The vectors in their vector fields differ only in length.

17. False. Note that $(x(0), y(0)) = (x(\pi), y(\pi)) = (0, 0)$. However, $(dx/dt, dy/dt) = (1, 1)$ at $t = 0$, and $(dx/dt, dy/dt) = (-1, -1)$ at $t = \pi$. For an autonomous system, the vector in the vector field at any given point does not vary as t varies. This function cannot be a solution of any autonomous system. (This function parameterizes a line segment in the xy-plane from $(1, 1)$ to $(-1, -1)$. In fact, it sweeps out the segment twice for $0 \le t \le 2\pi$.)

18. True. For an autonomous system, the rates of change of solutions depend only on position, not on time. Hence, if a function $(x_1(t), y_1(t))$ satisfies an autonomous system, then the function given by

$$(x_2(t), y_2(t)) = (x_1(t + T), y_1(t + T)),$$

where T is some constant, satisfies the same system.

19. False. The point $(0, 0)$ is an equilibrium point, so the Uniqueness Theorem guarantees that it is not on the solution curve corresponding to $(1, 0)$.

20. False. From the Uniqueness Theorem, we know that the solution curve with initial condition $(1/2, 0)$ is trapped by other solution curves that it cannot cross (or even touch). Hence, $x(t)$ and $y(t)$ must remain bounded for all t.

21. False. These solutions are different because they have different values at $t = 0$. However, they do trace out the same curve in the phase plane.

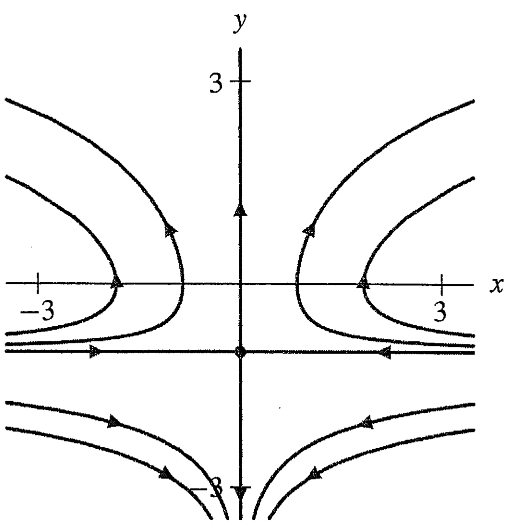
(c) If $(x(0), y(0)) = (1, 0)$, then we must solve the simultaneous equations

$$\begin{cases} k_2 e^{k_1} = 1 \\ -1 + k_1 = 0. \end{cases}$$

Hence, $k_1 = 1$, and $k_2 = 1/e$. The solution to the initial-value problem is

$$(x(t), y(t)) = \left(e^{-1}e^{-t+e^t}, -1 + e^t\right) = \left(e^{e^t-t-1}, -1 + e^t\right).$$

(d)

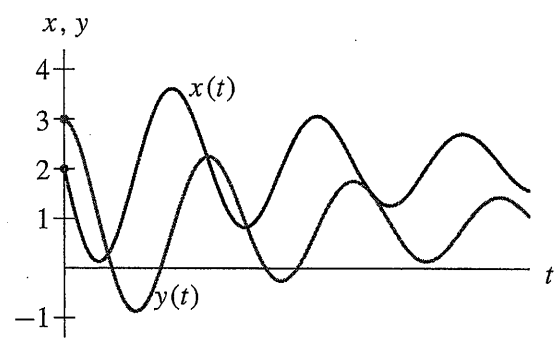


28. If x_1 is a root of $f(x)$ (that is, $f(x_1) = 0$), then the line $x = x_1$ is invariant. In other words, given an initial condition of the form (x_1, y), the corresponding solution curve remains on the line for all t. Along the line $x = x_1$, $y(t)$ obeys $dy/dt = g(y)$, so the line $x = x_1$ looks like the phase line of the equation $dy/dt = g(y)$.

Similarly, if $g(y_1) = 0$, then the line $y = y_1$ looks like the phase line for $dx/dt = f(x)$ except that it is horizontal rather than vertical.

Combining these two observations, we see that there will be vertical phase lines in the phase portrait for each root of $f(x)$ and horizontal phase lines in the phase portrait for each root of $g(y)$.

29.



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