Test 2: Differential Equations

Math 341 Fall 2010 — Friday November 19

 — 2:30pm-3:25pm

Name: ______________________________

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, limited-notes*, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

*You may use a one-sided 8.5" by 11" "cheat sheet" which must be stapled to the exam.

Offer: If there is a formula or piece of information that you feel that you need in order to solve a problem, I will provide it to you at a non-negotiable rate of at least a one point deduction.

Pledge: I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		15
2		15
3		20
BONUS		4
Total		**50**

1. *[15 points total.]* **Linear Systems of Differential Equations, Trace-Determinant Plane, Bifurcation.** VISUAL & ANALYTIC.

Consider $\dfrac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} \alpha & -\alpha/2 \\ 1 & -1 \end{bmatrix}$ and α is a known real-valued parameter. Recall that the eigenvalues of the matrix A are given by $\lambda = \dfrac{T \pm \sqrt{T^2 - 4D}}{2}$ where T is the trace of the matrix A and D is the determinant of matrix A.

1(a) *[5 points]*. Compute the trace T and determinant D of matrix A for all values of α. Show that the relationship between the trace T and determinant D is $2D + T = -1$ regardless of the value of α.

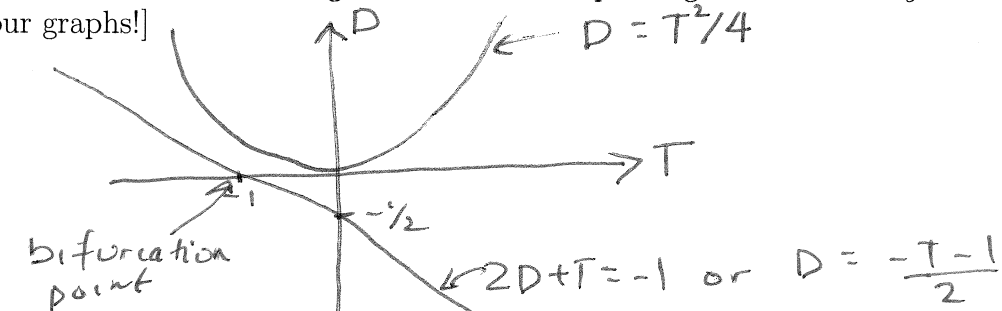
$$T = \alpha - 1 \Rightarrow T + 1 = \alpha$$

$$D = -\alpha + \frac{\alpha}{2}$$

$$= -\frac{\alpha}{2} \Rightarrow 2D = -\alpha = -(T+1) = -T - 1$$

$$\boxed{2D + T = -1}$$

1(b) *[5 points]*. Use your answer from (a) to sketch a graph in the trace-determinant plane depicting the relationship between the trace T and determinant D for the given matrix A as α changes. On the same axes, sketch the standard graph in the trace-determinant plane which separates the occurrence of real eigenvalues from complex eigenvalues for *any* matrix. [HINT: Label your graphs!]



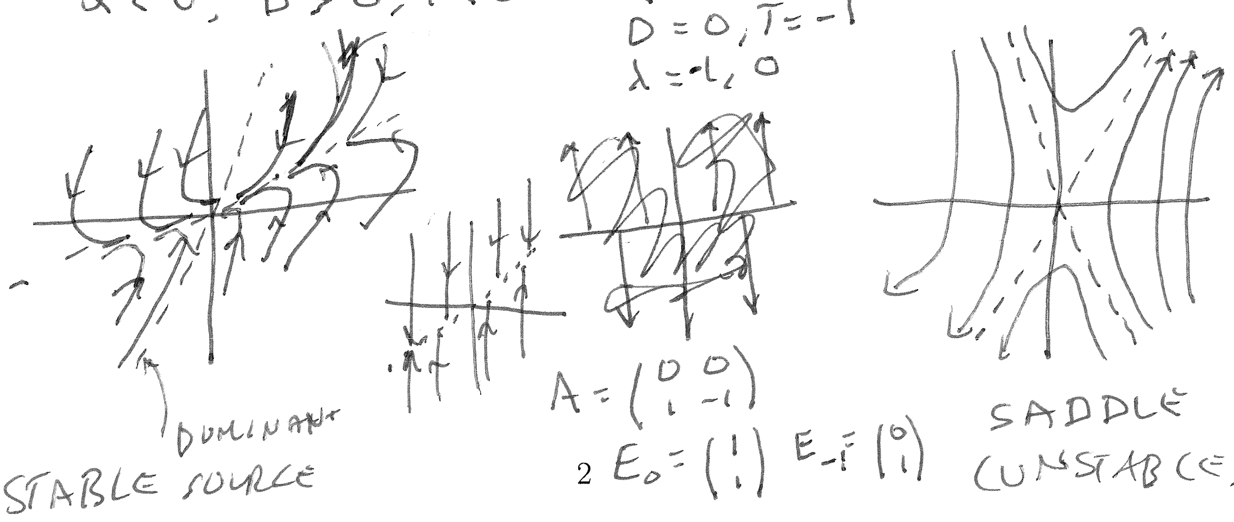
1(c) *[5 points]*. Does the qualitative nature of the phase portrait (and equilibrium at the origin) change as α varies? If so, **give all the bifurcation values of α, classify the equilibrium point at the origin** for values of α (greater than, less than and equal to the bifurcation value(s)) and **provide reasonable sketches of the phase portrait(s))** in each case in the space below.

$D = 0 = -\frac{\alpha}{2} \Rightarrow \alpha = 0 = \alpha_B$

$D < 0, \alpha > 0$

$\alpha < 0, D > 0, T < 0$

$\alpha = 0$
$D = 0, T = -1$
$\lambda = -1, 0$



2. *[15 points total.]* **Linearization, Hamiltonian function, Gradient function.** ANALYTIC, VERBAL & VISUAL.

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box. **For Full Credit you must write a full sentence explaining the reason for your choice of TRUE or FALSE.**

Consider the non-linear system

$$\frac{dx}{dt} = \alpha x - y + y^3 = f(x,y)$$
$$\frac{dy}{dt} = x + \alpha y + x^2 = g(x,y)$$

where α is a known real-valued parameter.

2(a) TRUE or FALSE? "A Hamiltonian function $H(x, y)$ for the given nonlinear system of ODEs exists."

$\boxed{\text{TRUE}}$

$f_x = \alpha$

$g_y = \alpha$

Hamiltonian Condition: $f_x = -g_y$

$\alpha = -\alpha$ true only if $\alpha = 0$, false when $\alpha \neq 0$

The Hamiltonian will only exist when $\alpha = 0$.

2(b) TRUE or FALSE? "A gradient function $G(x, y)$ for the nonlinear system of ODEs exists."

$\boxed{\text{FALSE}}$

$f_y = -1 + 3y^2$

$g_x = 1 + 2x$

Gradient condition: $f_y = g_x$

$-1 + 3y^2 = 1 + 2x$

Since the gradient condition does not hold for all x, y this system does NOT have a $G(x,y)$.

2(c) TRUE or FALSE? "The origin of the phase portrait for this nonlinear system will look like a center (a series of concentric circles) for all values of α."

$\boxed{\text{FALSE}}$

When $\alpha = 0$ is the only time the origin will look like a center.

$$J = \begin{pmatrix} \alpha & -1+3y^2 \\ 1+2x & \alpha \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \quad \lambda^2 - \alpha\lambda + \alpha^2 + 1$$

$T = 2\alpha$

$D = \alpha^2 + 1$

~~so~~ when $T = 0 = \alpha$ λ will be complex and origin will be centers

3. *[20 points total.]* **Linear Systems of Differential Equations, Matrix Exponential, General Solution.** ANALYTIC & VERBAL.

Consider the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x_0}$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Usually we write the general solution as $\vec{x} = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$ where λ_i and $\vec{v_i}$ are eigenvalues and eigenvectors of matrix A. However, the solution can also be written as $\vec{x}(t) = e^{At}\vec{x}_0$, which we will call the **matrix exponential solution**.

The goal of this problem is to show that the general solution to the given initial value problem (which we will call the **general eigenvector solution**) can be represented using the matrix exponential e^{At}.

Recall from Calculus that $\frac{d}{dt}e^{\Box t} = \Box e^{\Box t}$ as long as $\frac{d}{dt}\Box = 0$.

Recall from Linear Algebra that if A is diagonalizable, then the matrix exponential $e^{At} = Se^{\Lambda t}S^{-1}$ where S is an $n \times n$ matrix whose columns consist of the n eigenvectors $\vec{v}_1, \ldots \vec{v}_n$ of A, and Λ is an $n \times n$ diagonal matrix with the corresponding eigenvalues along the diagonal.

$$e^{At} = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \ldots & \vec{v_n} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \ldots & \vec{v_n} \end{bmatrix}^{-1}$$

NOTE: For this problem you can assume that A is a 2×2 diagonalizable matrix of the form $\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$ (where p and q are known fixed numbers) and A has two eigenvalues λ_1 and λ_2 with associated eigenvectors $\vec{v_1} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$.

3(a) *[3 points]*. Show that the **the general eigenvector solution** satisfies the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$.

$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

$\frac{d\vec{x}}{dt} = \lambda_1 c_1 e^{\lambda_1 t}\vec{v}_1 + \lambda_2 c_2 e^{\lambda_1 t}\vec{v}_2 = LHS$

$A\vec{x} = c_1 e^{\lambda_1 t}A\vec{v}_1 + c_2 e^{\lambda_2 t}A\vec{v}_2 = RHS$

since $A\vec{v}_1 = \lambda\vec{v}_1$ and $A\vec{v}_2 = \lambda\vec{v}_2$

$LHS = RHS$

3(b) *[3 points]*. Show that in order for **the general eigenvector solution** to satisfy the given initial condition $\vec{x}(0) = \vec{x_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, the linear system $\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ must be solved to find the unknown constants c_1 and c_2. (HINT: What's a restriction involving λ_1 and λ_2 which must be satisfied?)

$\vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

This matrix is only invertible if

$Det = \lambda_1 - \lambda_2 \neq 0$

3(c) *[6 points].* Show that **the matrix exponential solution** $\vec{x}(t) = e^{At}\vec{x}_0$ satisfies the given initial value problem. [HINT: for a given solution to satisfy an initial value problem what must be true?]

$$\vec{x} = e^{At}\vec{x}_0$$

$$\frac{d\vec{x}}{dt} = A\underbrace{e^{At}\vec{x}_0}_{\vec{x}} = A\vec{x} \Rightarrow \text{The DE is satisfied}$$

$$t=0,\ \vec{x} = \vec{x}_0$$

$$\vec{x} = e^{A\cdot 0}\vec{x}_0$$
$$= I\vec{x}_0$$
$$= \vec{x}_0 \Rightarrow \text{The IC is satisfied}$$

3(d) *[8 points].* Use your results from (a),(b) and (c) to show that **the matrix exponential solution** is identical to **the general eigenvector solution** for the given initial value problem $\dfrac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x_0}$ where A has two eigenvalues λ_1 and λ_2 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$. (HINT: What's a restriction involving λ_1 and λ_2 which must be satisfied?) EXPLAIN YOUR ANSWER(S) and SHOW ALL YOUR WORK.

$$e^{At}\vec{x}_0 \overset{?}{=} c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} \overset{?}{=} c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \overset{?}{=} \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1\lambda_1 e^{\lambda_1 t} + c_2\lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

$$\begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \overset{?}{=}$$

$$\begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1\lambda_1 e^{\lambda_1 t} + c_2\lambda_2 e^{\lambda_2 t} \end{pmatrix} \overset{?}{=} \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1\lambda_1 e^{\lambda_1 t} + c_2\lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

LHS = RHS ✓

Note $\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$ is only invertible iff $\lambda_1 \neq \lambda_2$

BONUS. *[4 points]*. Find the (matrix exponential) solution to the initial value problem $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and simplify it into the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

CHECK YOUR ANSWER!

HINT: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$T = -3$

$\Delta = 2$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\lambda = -1, -2$

$$\vec{x}(t) = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \frac{1}{-1} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{pmatrix}$$

$t = 0$, $\vec{x}(0) = \begin{pmatrix} 3-2 \\ -3+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓ (IC)

$$\vec{x}' = \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ 3e^{-t} - 8e^{-2t} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ -6e^{-t} + 4e^{-2t} + 9e^{-t} - 12e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ 3e^{-t} - 8e^{-2t} \end{pmatrix}$$

LHS = RHS ✓ (DE)