Test 1: Differential Equations

Math 341 Fall 2009

Wednesday October 21
2:30pm-3:25pm

Name: Key

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		15
2		15
3		20
BONUS		5
Total		**50**

1. *[15 points total.]* **Existence and Uniqueness, Families of Solutions, Singular Solutions, Separation of Variables.** ANALYTIC & VERBAL.
Consider the initial value problem

$$\frac{dy}{dt} = 3y^{2/3}, \quad y(0) = A, \text{where } A \text{ is any real number.}$$

1(a) *[5 points].* Show that the solution to this initial value problem has the form $y(t) = (t + A^{1/3})^3$.

$$\frac{dy}{dt} = 3y^{2/3}$$

$$\int y^{-\frac{2}{3}} dy = \int 3\, dt$$

$$3y^{1/3} = 3t + C$$

$$y^{1/3} = t + D$$

$$t = 0, y = A \Rightarrow A^{1/3} = D \Rightarrow$$

$$y^{1/3} = t + A^{1/3}$$

$$y = (t + A^{1/3})^3$$

OR

$$\text{IC: } t = 0,\ y = (0 + A^{1/3})^3 = A \checkmark$$

$$\text{DE} \quad \frac{d}{dt}(t + A^{1/3})^3 = 3(t + A^{1/3})^2 = 3 \cdot \left[(t + A^{1/3})^3\right]^{2/3} = 3y^{2/3} \checkmark$$

1(b) *[5 points].* Is this the only possible solution of the given initial value problem that exists? (Remember to consider all values of A.) EXPLAIN YOUR ANSWER.

If $A \neq 0$ $f(t,y) = 3y^{2/3}$ AND $f_y(t,y) = 2y^{-1/3}$ are continous at all places $(0, A)$ which implies unique solutions will exist.

If $A = 0$ f will be continous but f_y will NOT be continous at $(0,0)$ so existence is guaranteed but not uniqueness.

$y = 0$ is ALSO a sol^n

1(c) *[5 points].* Discuss how (and/or whether!) the Existence and Uniqueness Theorem can be used to explain the existence of solutions to the given initial value problem AND the uniqueness of these solutions. (Remember to consider all values of A.)

$A \neq 0$, E&UT guarantees existence of unique solutions.

$A = 0$, existence is guaranteed, but not uniqueness. It turns out that the solution is NOT unique at this point, but this is NOT due to the theorem.

2. *[15 points total.]* **Slope Fields of DEs, Equilibria, Homogeneous and Non-homogeneous DEs.** ANALYTIC, VERBAL & VISUAL.
Multiple Choice (Clicker-like) Questions. Cicle the letter of your answer choice and then explain your answer in the space provided. Your explanation of your answer choice is worth FOUR TIMES as much as the choice itself.

2(a) Conside the differential equation $\dfrac{dg}{dz} = ag + b\cos(cz)$, where a, b and c are positive parameters. What will the long-term behavior of the system be?

A. The solution will grow exponentially.

B. The solution will converge to an equilibrium.

C. The solution will oscillate.

D. Different behaviors are possible depending on the values of a, b and c.

$g(z) = C_1 e^{az} + C_2 \cos(cz) + C_3 \sin(cz)$

As $z \to \infty$, $e^{az} \to \infty$ and dominate the trig terms.

2(b) What is the equilibrium value of $\dfrac{da}{db} = a + 3e^b$?

A. The system is in equilibrium when $a = -3e^b$.

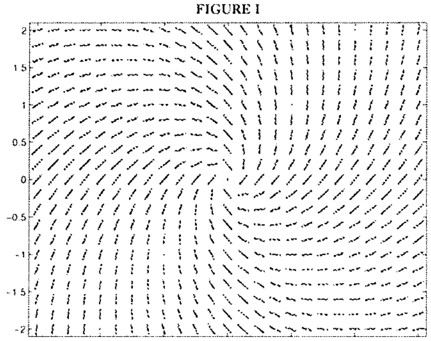
B. The system is in equilibrium when $b = \ln\left(\dfrac{-a}{3}\right)$

C. Both A. and B. are true.

D. This equation has no equilibrium value.

Autonomous DEs have equilibrium values. $a' = a + 3e^b$ is not autonomous, so it doesn't have an equilibrium

FIGURE I



2(c) Which of the following differential equations would generate the given slopefield in Figure I?

A. $y' = 1/x$

B. $y' = 1/y$

C. $y' = \dfrac{x+y}{x-y}$

D. $y' = \dfrac{x-y}{x+y}$

Notice at $(1,1), (2,2), (-1,-1), (-2,-2)$ when $y = x$, there is no slope! This mean $(y-x)$ must be in denominator. Slopefield clearly depends on both x and y since the slopes are not remaining parallel as we move vertically or horizontally

BONUS. Find the general solution of $\frac{dx}{dt} = x + 3y$, $\frac{dy}{dt} = y$.

~~DE~~ DECOUPLED!

$y(t) = Ae^t$

$\frac{dx}{dt} = x + 3Ae^t$ non-homogeneous DE

Pick: $x = \underbrace{Be^t}_{y_h} + \underbrace{Dte^t}_{y_p}$

$\frac{dx}{dt} = Be^t + De^t + Dte^t$

$LHS = \frac{dx}{dt} - x = (Be^t + De^t + Dte^t) - Be^t - Dte^t$

$= De^t$

$RHS = 3Ae^t$

$3A = D$

$x(t) = Be^t + Ate^t$

$y(t) = Ae^t$

$\vec{x} = Be^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Ate^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

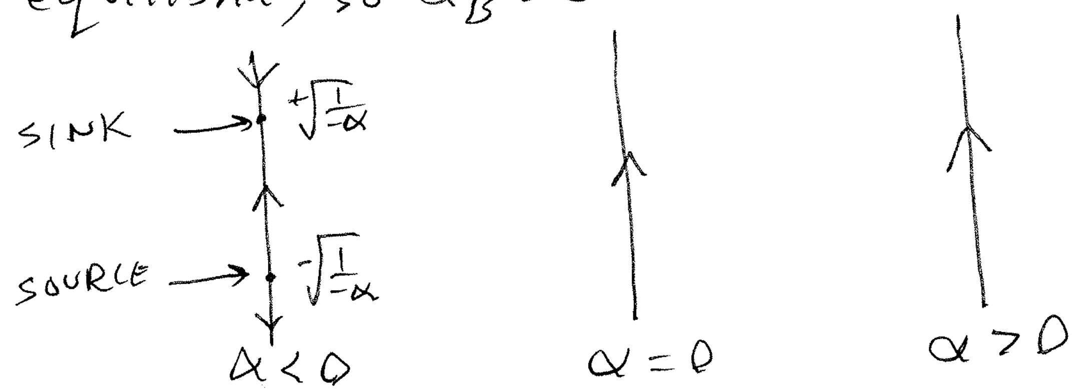
3. *[20 pts. total]* **Phase Lines, Equilibria, Bifurcations, Geometric Representations.**
VISUAL & ANALYTIC.
Consider the following differential equation: $\frac{dy}{dt} = \alpha y^2 + 1$, where α is a parameter.

3(a) *[6 pts]* What are the equilibrium values of y? Identify them as y^*.

$$\alpha y^2 + 1 = 0 \Rightarrow y^* = \pm\sqrt{\frac{-1}{\alpha}}$$

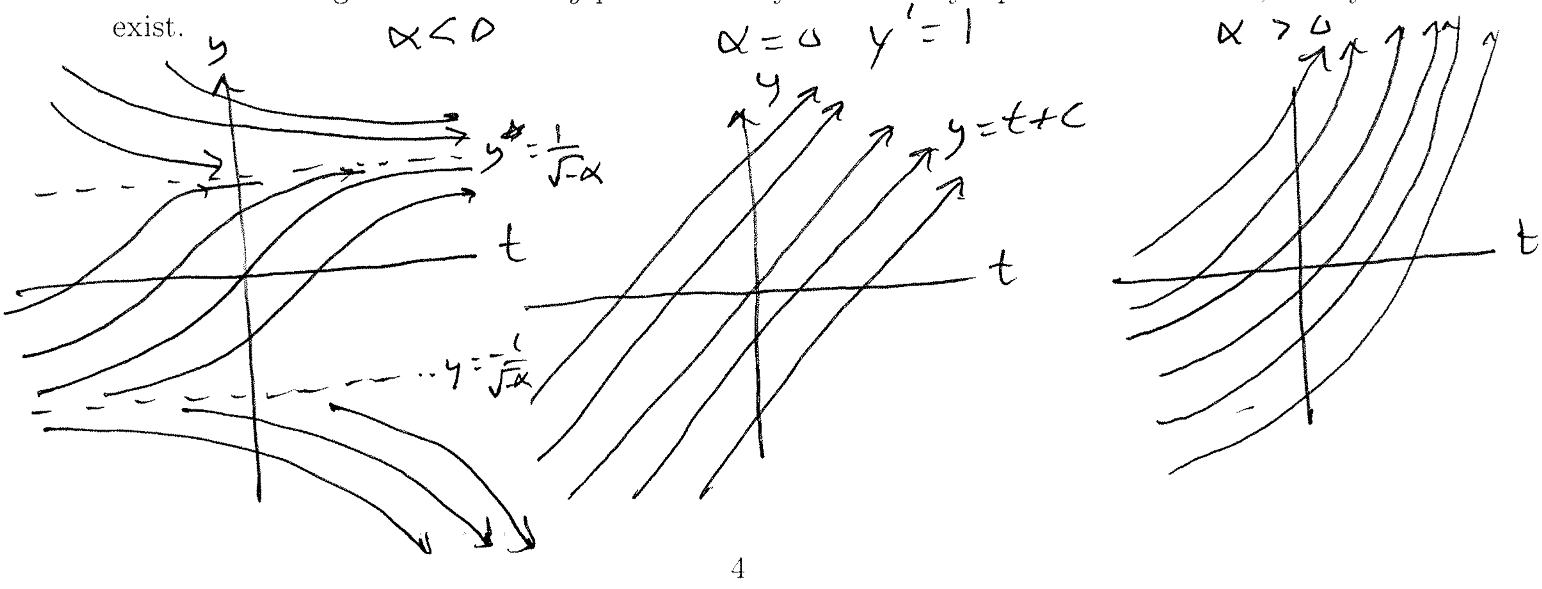
3(b) *[3 points]* Is there a bifurcation value for the parameter α? If so, identify it as α_B and draw phase lines corresponding to the cases where $\alpha < \alpha_B$, $\alpha = \alpha_B$ and $\alpha > \alpha_B$. Indicate locations (and values) of y^* on your phase lines.

Clearly when $\alpha = 0$ there will be a change in the equlibria, so $\alpha_B = 0$

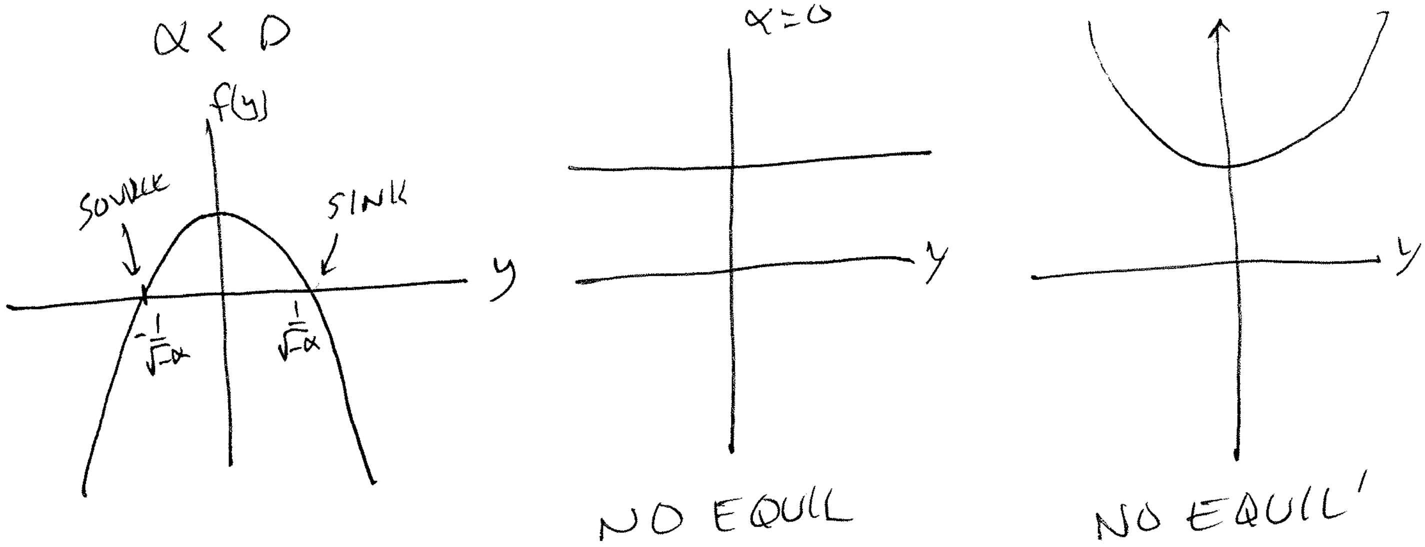


Find $y^* = y^*(\alpha)$
where $f(y^*; \alpha) = 0$
$y^* = \pm\sqrt{\frac{1}{\alpha}}$
Use this value of y to find α
so that
$f'(y^*; \alpha) = 0$
$2\alpha \pm \sqrt{\frac{1}{\alpha}} = 0$
$\boxed{\alpha = 0}$

3(c) *[3 points]* Choose a particular value of α of your choice and draw a phase portrait of solutions to the given DE in the ty-plane. Clearly indicate any equilibrium solutions, if they exist.



3(d) *[3 points]* Considering the differential equation as $\frac{dy}{dt} = f(y;\alpha) = \alpha y^2 + 1$, sketch graphs of $f(y)$ versus y corresponding to the cases where $\alpha < \alpha_B$, $\alpha = \alpha_B$ and $\alpha > \alpha_B$. Clearly indicate any equilibrium solutions, if they exist.



3(e) *[5 points]* Draw a bifurcation diagram for the differential equation in the αy-plane. Indicate clearly where sinks, sources and nodes occur, if they exist.