# Differential Equations

Math 341 Fall 2009 — MWF 2:30-3:25pm Fowler 110

 — `http://faculty.oxy.edu/ron/math/341/09/`

## *Worksheet 21*: Wednesday November 4

**TITLE** The Trace-Determinant Plane
**CURRENT READING** Blanchard, 3.7

**Homework Assignments due Friday November 13**
Section 3.7: 1, 6.
Section 5.1: 3, 4, 5, 18, 21.
Section 5.2: 3, 4, 16.

**SUMMARY**
We shall summarize all the possible equilibria one can get with a 2x2 linear system of ODEs into one big picture!

## 1. Summarizing The Possibilities

Given a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the characteristic polynomial is $(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \text{tr(A)}\lambda + \det(\text{A}) = 0$.

GROUPWORK

Your goal is to match the case # in the left column with the description of its critical point on the right (the list now is jumbled).

| | |
|---|---|
| **CASE 1**: Real $\lambda$, $\lambda_1\lambda_2 < 0$ | **A** Center |
| **CASE 2**: Real $\lambda$, $\lambda_1 \& \lambda_2 < 0$ | **B** Spiral Source |
| **CASE 3**: Real $\lambda$, $\lambda_1 \& \lambda_2 > 0$ | **C** (Stable) Node |
| **CASE 4**: Real $\lambda$, $\lambda_1 = \lambda_2 > 0$ | **D**(Unstable) Node |
| **CASE 5**: Real $\lambda$, $\lambda_1 = \lambda_2 < 0$ | **E** Saddle |
| **CASE 6**: Complex $\lambda$, $\text{Re}(\lambda) > 0$ | **F** Spiral Sink |
| **CASE 7**: Complex $\lambda$, $\text{Re}(\lambda) < 0$ | **G** Sink |
| **CASE 8**: Complex $\lambda$, $\text{Re}(\lambda) = 0$ | **H** Source |

Run the CD-Rom from our textbook and select `LinearPhasePortaits`. Use the slide bars to obtain different values of $a$, $b$, $c$ and $d$ and the different kinds of eigenvalues recorded above in the Cases. Record your results in the table below.

| CASE # | a | b | c | d | $\lambda_1$ | $\lambda_2$ | Description |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | | | | | | | |
| 3 | | | | | | | |
| 4 | | | | | | | |
| 5 | | | | | | | |
| 6 | | | | | | | |
| 7 | | | | | | | |
| 8 | | | | | | | |

For more details, see the handout from **Edwards and Penney,** *Differential Equations*, **3rd Edition, Prentice Hall: 2004.**

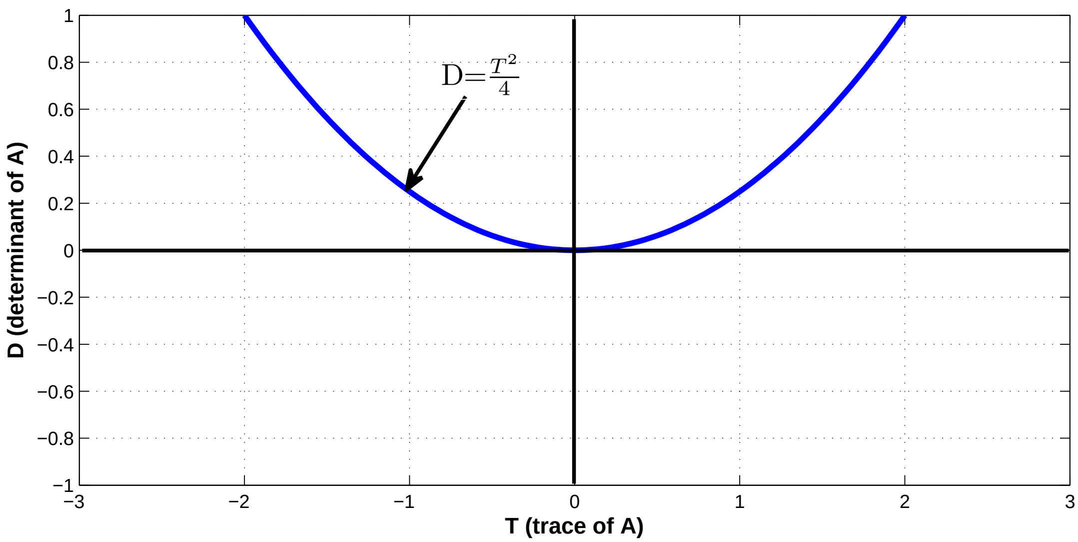
## 2. The Trace-Determinant Plane

Recall that the eigenvalues of a 2x2 matrix are given by the rootsof the polynomial $p(\lambda) = \lambda^2 - \text{tr}(A) + \det(A) = 0$.

It's also true that the trace of $A$, denoted $\text{tr}(A)$ is equal to the **sum** of the eigenvalues $\lambda_1 + \lambda_2$. Let's use the symbol $T$ for $\text{tr}(A)$. The determinant of $A$, denoted $\det(A)$ is equal to the **product** of the eigenvalues $\lambda_1\lambda_2$. Let's use the symbol $D$ for $\det(A)$.

Then we know that the eigenvalues are given by the solutions to $\lambda^2 - T\lambda + D = 0$, or $\lambda = \dfrac{T \pm \sqrt{T^2 - 4D}}{2}$.

In other words, the condition on whether we will have real, complex or repeated eigenvalues depends on the behavior of the discriminant $\Gamma = T^2 - 4D$. See the figure drawn below. This is known as the **Trace-Determinant Plane**




This graph is an example of a parameter plane. As the matrix $A$ changes it has different values of $T$ and $D$ and the linear system $\dfrac{d\vec{x}}{dt} = A\vec{x}$ corresponding to that matrix will be located at a diferent location in $(T, D)$-space.

**Exercise**

(1) What kind of phase portraits will exist in $(T, D)$-space along the $D$ axis?

(2) What about the $T$-axis?

(3) What kind of phase portraits occur along the curve $D = \frac{T^2}{4}$?

(4) What happens as one moves from the region just above the $T$-axis $(D > 0)$ to just below the $T$-axis $(D > 0)$? Does it matter if $T >)$ or $T < 0$?

(5) What kinds of solutions exist in the region above the parabola $D = \frac{T^2}{4}$?