Differential Equations

Math 341 Fall 2009 MWF 2:30-3:25pm Fowler 110

 http://faculty.oxy.edu/ron/math/341/09/

Worksheet 20: Monday November 2

TITLE Second Order Linear ODEs
CURRENT READING Blanchard, 3.6

Homework Assignments due Friday November 6
Section 3.5: 3, 4, 9, 10, 17, 18.
Section 3.6: 4, 5, 16, 33, 38.

SUMMARY
The joys of harmonic motion! We shall more closely examine the standard second order constant-coefficient ODE $y'' + ay' + by = 0$ now that we have completed the analysis of the analogous first order system of 2 linear ODEs.

1. Recalling 2nd Order Linear Systems of ODEs

Recall that the 2^{nd} order constant coefficient ODE $y'' + py' + qy = 0$ where p and q are real-numbered constants can be written as the linear system

$$\frac{dy}{dt} = v; \ \ \frac{dv}{dt} = -pv - qy.$$

which can also be written as

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{x} \qquad \text{where } \vec{x} = \begin{bmatrix} y \\ v \end{bmatrix} \text{ or } \begin{bmatrix} y \\ y' \end{bmatrix}$$

Clearly this is a special case of the linear systems of ODEs we have been examining for quite awhile that look like $\frac{d\vec{x}}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$ where a,b, c and d are constants.

RECALL
On *Worksheet 12* we had shown that making the guess that $y(t) = e^{rt}$ into $y'' + py' + qy = 0$ produces a polynomial $r^2 + pr + q = 0$ whose roots $r1$ and r_2 lead to the general solution $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

EXAMPLE
We can use the general solution of the 2nd Order Linear ODE $y'' + py + qy = 0$ to write down the general solution to $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{x}$

Does this look familiar? What is the connection to the work we have done before?

Exercise

Show that the characteristic polynomial for $\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$ is $\lambda^2 + p\lambda + q = 0$.

So, the solutions to $r^2+pr+q = 0$ and $\lambda^2+p\lambda+q = 0$ are ________________________________.

Also, if we know the solutions to the characteristic polynomial, then we know that the eigenvectors of matrix A are ______________ and __________________.

What this means is that we can use our recently completed analysis of linear systems of ODEs (From Sections 3.2 through 3.5) to look at damped harmonic motion.

2. Damped Harmonic Oscillator

It turns out that the equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

represents the displacement $y(t)$ of a damped mass-spring system where m is the mass, b is the damping constant and k is the spring constant. This type of motion is known as **damped harmonic motion.**

By making the guess $y = e^{rt}$ we obtain the following quadratic equation $mr^2 + br + k = 0$ as the characteristic polynomial for damped harmonic motion.

EXAMPLE

(a) Let's write the ODE for damped harmonic motion in the form $\frac{d\vec{x}}{dt} = A\vec{x}$ where $\vec{x} = \begin{bmatrix} y \\ y' \end{bmatrix}$.

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \vec{x}$$

(b) Let's find the solution of the characteric polynomial for damped harmonic motion.

We already know that the discriminant Γ of this quadratic equation will be either positive, negative or zero and this will have immediate implications for what the solutions $y(t)$ as well as the phase portrait of the linear system $\frac{d\vec{x}}{dt} = A\vec{x}$.

GroupWork

Fill out the following table and use the program `LinearPhasePortraits` on the computers in `Start--> My Computer-->S ('stuserver')--> Math Courses--> Math 341-->Fall 2009` to come up with examples which reflect the three cases given below of **overdamped**, **critically damped** and **under-damped** harmonic oscillations.

	$\Gamma < 0$	$\Gamma = 0$	$\Gamma > 0$
Conditions on m,b and k			
eigenvalues are (real or complex)			
number of eigenvalues (0, 1 or 2)			
oscillations? (yes or no)			
damping? (over-, under-, critically-)			
phase portrait description			

Exploration

Check out the website `http://www.math.ualberta.ca/ ewoolgar/java/Hooke/Hooke.html` or Google "damped harmonic oscillation applet" to find a web-based demonstration which illustrates damped harmonic motion. There's also a link from the Math 341 class website in the Resources section. **What patterns do you observe?**