Differential Equations

Math 341 Fall 2009 MWF 2:30-3:25pm Fowler 110

 http://faculty.oxy.edu/ron/math/341/09/

Worksheet 11: Monday October 5

TITLE Geometry of First Order Systems of ODEs
CURRENT READING Blanchard, 2.2

Homework Assignments due Friday October 9
Chapter 1 Review: 3, 4, 6, 10, 11, 12, 29, 47.
Section 2.1: 1, 2, 3, 5, 7, 10.
Section 2.2: 7, 8, 10, 24, 25.

SUMMARY
We will learn how to create the beautiful pictures which can result when one does quantitative analysis on systems of ODEs (phase portraits).

1. Vector Notation and Vector Fields

Let $\vec{x} = \begin{bmatrix} R(t) \\ F(t) \end{bmatrix}$ and $\frac{d\vec{x}}{dt} = \begin{bmatrix} R'(t) \\ F'(t) \end{bmatrix}$, the Lotka-Volterra equations can be re-written as:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} aR - bRF \\ cF + dRF \end{bmatrix} = \vec{P}(\vec{x}, t)$$

Note that the above $\vec{P}$ is a vector function of a vector input, in the case of Lotka-Volterra $P : \mathbb{R}^3 \to \mathbb{R}^2$

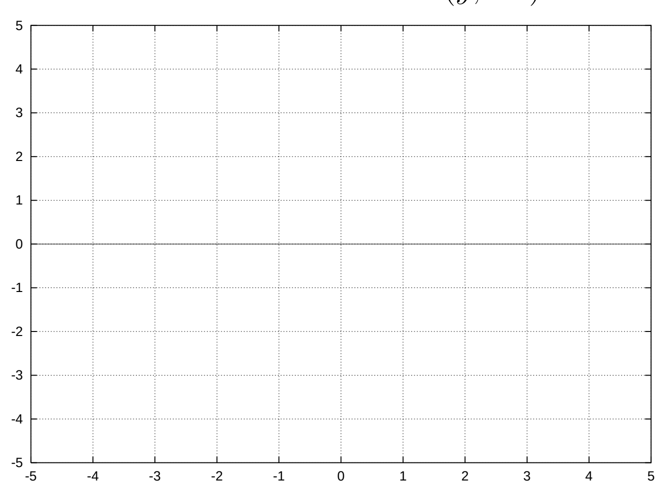
DEFINITION: **fixed point of linear system of ODEs**

A fixed point or equilibrium point or stationary point $\vec{x_0}$ of the system $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, t)$ is a point at which $\vec{F}(\vec{x_0}) = \vec{0}$.

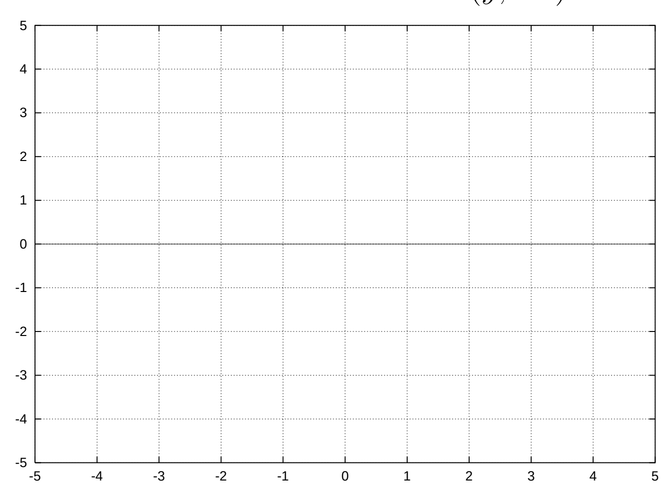
RECALL
We can visualize vector functions using **vector fields**. Consider the function $\vec{F}(x, y) = \begin{bmatrix} y \\ -x \end{bmatrix}$. Sketch the vector field in the axes to the left, below. Generally, we normalize the vectors to all have the same magnitude and produce something that is called a **direction field**. It looks exactly like a **slope field**, except the "lineal elements" have little arrows on them.

EXAMPLE Let's draw the vector field and direction field for $\vec{F}(\vec{x}) = (y, -x)$.

Vector Field for $\vec{F} = (y, -x)$



Direction Field for $\vec{F} = (y, -x)$



2. Direction Fields and 1st Order Linear Systems of ODEs

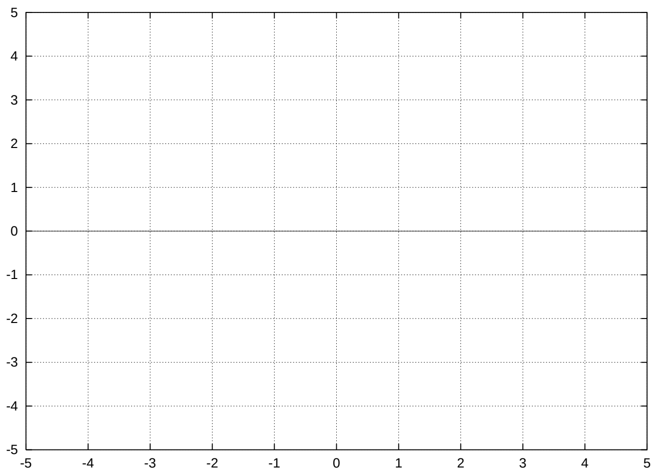
Consider the system of ODES

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x$$

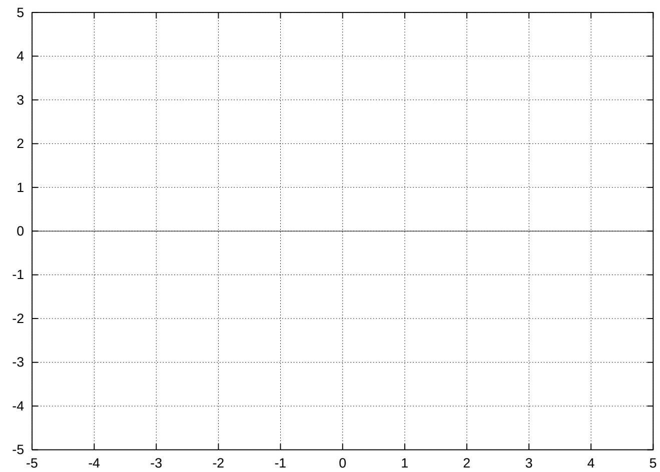
GROUPWORK

Draw a solution curve that starts at $x(0) = 0, \quad y(0) = 1$ on the axes to the left, and on the right draw solution curves of $x(t)$ and $y(t)$ on the same axes.

Solution Curve for $\vec{x}(0) = (0, 1)$



$x(t)$ and $y(t)$ versus t



Q: Does the system have any equilibria?
A: ____

3. Nullclines

Consider the general 2-D system of ODEs

SYSTEM A

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) = y \\ \frac{dy}{dt} &= g(x,y) = -x \end{aligned}$$

DEFINITION: **nullcline**

A curve along which a derivative (with respect to the independent variable) is zero is said to be a nullcline. In other words, one of the variables will be constant, while the other variable varies with respect to t. An x-nullcline is a set of points for which x is constant (i.e. $\frac{dx}{dt} = 0$). Algebraically, $\{(x, y) \mid \frac{dx}{dt} = 0\}$. A y-nullcline is a set of points for which y is constant (i.e. $\frac{dy}{dt} = 0$). In this case, $\{(x, y) \mid \frac{dy}{dt} = 0\}$.

Q: What are the nullclines of System A?
A: ____

4. Phase Portrait

The phase portrait of a system is a diagram showing the set of solution curves in the phase plane of a system of ODEs.

SYSTEM B

$$\frac{dx}{dt} = 5x$$
$$\frac{dy}{dt} = -y$$

Exercise

Sketch the phase portrait of the system. This means sketching the nullclines, clearly indicating any fixed points and including several solution curves in the phase plane.

SYSTEM C

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = 4x - 2y$$

Exercise

Sketch the phase portrait of the system. This means sketching the nullclines, clearly indicating any fixed points and including several solution curves in the phase plane.

SYSTEM D

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\frac{k}{m}x\end{aligned}$$

This system should look familar, or if it doesn't perhaps this differential equation is

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{1}$$

The above equation is the equation of motion for **harmonic motion** and has the known solutions $x(t) = A\cos(\omega t) + B\sin(\omega t)$ where $\omega^2 = \dfrac{k}{m}$ and ω is the frequency of the motion and $\dfrac{2\pi}{\omega}$ is the period of the oscillation of a mass on a string (with no damping).

What's the relationship between Equation 1 and System D? Is there a way to convert one into the other?

GroupWork

Use technology to sketch phase portraits of System D for various values of the ratio k/m.