EXERCISES FOR SECTION 3.4

1. Using Euler's formula, we can write the complex-valued solution $\mathbf{Y}_c(t)$ as

$$\begin{aligned}\mathbf{Y}_c(t) &= e^{(1+3i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t e^{3it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t(\cos 3t + i\sin 3t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\ &= e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + ie^t\begin{pmatrix} 2\sin 3t + \cos 3t \\ \sin 3t \end{pmatrix}.\end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\mathrm{re}}(t) = e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\mathrm{im}}(t) = e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + k_2 e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

2. The complex solution is

$$\mathbf{Y}_c(t) = e^{(-2+5i)t}\begin{pmatrix} 1 \\ 4-3i \end{pmatrix},$$

so we can use Euler's formula to write

$$\begin{aligned}\mathbf{Y}_c(t) &= e^{(-2+5i)t}\begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\ &= e^{-2t}e^{5it}\begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\ &= e^{-2t}(\cos 5t + i\sin 5t)\begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + ie^{-2t}\begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.\end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\mathrm{re}}(t) = e^{-2t}\begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\mathrm{im}}(t) = e^{-2t}\begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^{-2t} \begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

3. **(a)** The characteristic equation is

$$(-\lambda)^2 + 4 = \lambda^2 + 4 = 0,$$

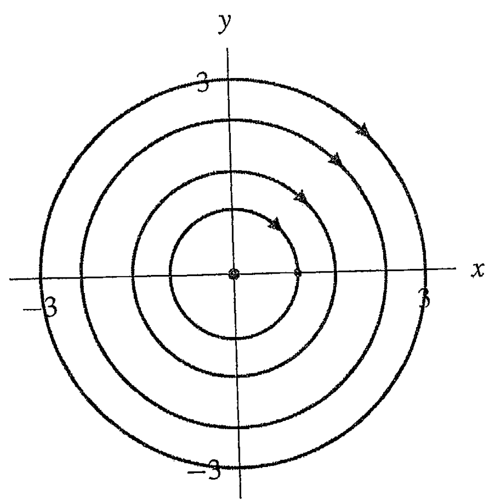
and the eigenvalues are $\lambda = \pm 2i$.

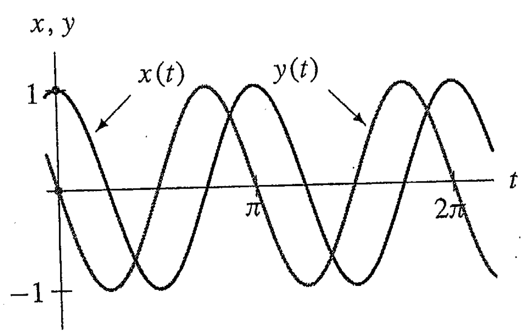
(b) Since the real part of the eigenvalues are 0, the origin is a center.

(c) Since $\lambda = \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At $(1, 0)$, the tangent vector is $(-2, 0)$. Therefore, the direction of oscillation is clockwise.

(e) According to the phase plane, $x(t)$ and $y(t)$ are periodic with period π. At the initial condition $(1, 0)$, both $x(t)$ and $y(t)$ are initially decreasing.





4. **(a)** The characteristic equation is

$$(2 - \lambda)(6 - \lambda) + 8 = \lambda^2 - 8\lambda + 20,$$

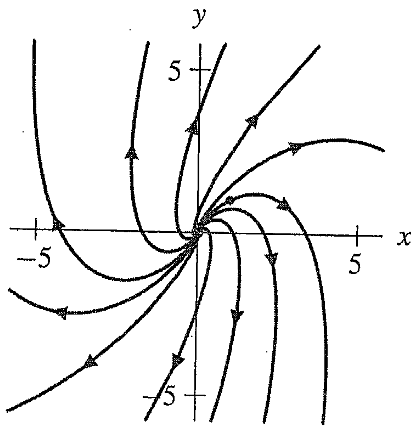
and the eigenvalues are $\lambda = 4 \pm 2i$.

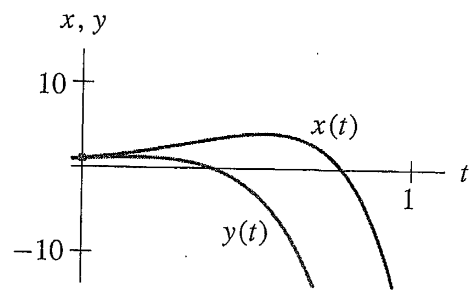
(b) Since the real part of the eigenvalues is positive, the origin is a spiral source.

(c) Since $\lambda = 4 \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At the point $(1, 0)$, the tangent vector is $(2, -4)$. Therefore, the solution curves spiral around the origin in a clockwise fashion.

(e) Since $d\mathbf{Y}/dt = (4, 2)$ at $\mathbf{Y}_0 = (1, 1)$, both $x(t)$ and $y(t)$ increase initially. The distance between successive zeros is π, and the amplitudes of both $x(t)$ and $y(t)$ are increasing.





5. (a) The characteristic polynomial is

$$(-3-\lambda)(1-\lambda)+15=\lambda^2+2\lambda+12,$$

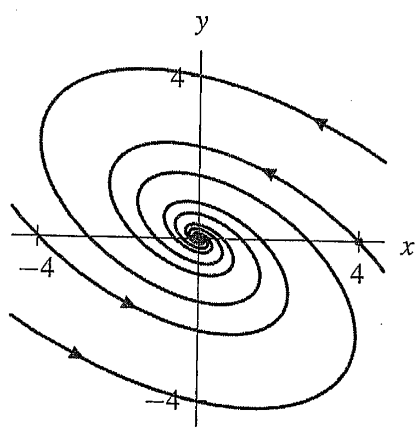
so the eigenvalues are $\lambda = -1 \pm i\sqrt{11}$.

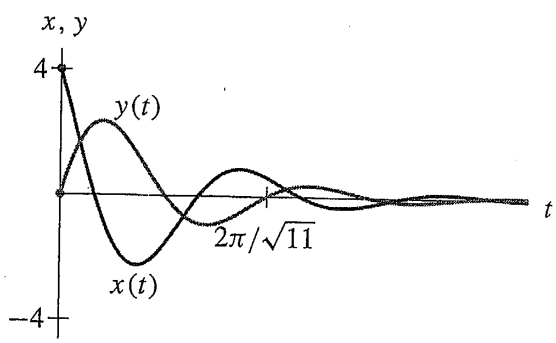
(b) The eigenvalues are complex and the real part is negative, so the origin is a spiral sink.

(c) The natural period is $2\pi/\sqrt{11}$. The natural frequency is $\sqrt{11}/(2\pi)$.

(d) At the point $(1, 0)$, the vector field is $(-3, 3)$. Hence, the solution curves must spiral in a counterclockwise fashion.

(e)





6. (a) The characteristic polynomial is

$$(-\lambda)(-1-\lambda)+4=\lambda^2+\lambda+4,$$

so the eigenvalues are $\lambda = (-1 \pm i\sqrt{15})/2$.

(b) The eigenvalues are complex and the real part is negative, so the origin is a spiral sink.

(c) The natural period is $2\pi/(\sqrt{15}/2) = 4\pi/\sqrt{15}$. The natural frequency is $\sqrt{15}/(4\pi)$.