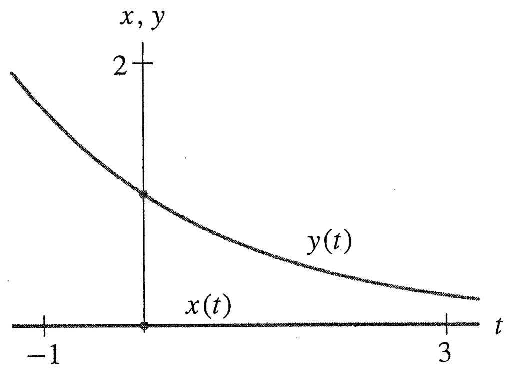


The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}(t)$.

(e) Since the method of eigenvalues and eigenvectors does not give us a second solution that is linearly independent from $\mathbf{Y}(t)$, we cannot form the general solution.

6. **(a)** The characteristic polynomial is

$$(5-\lambda)(-\lambda) - 36 = 0,$$

and therefore the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 9$.

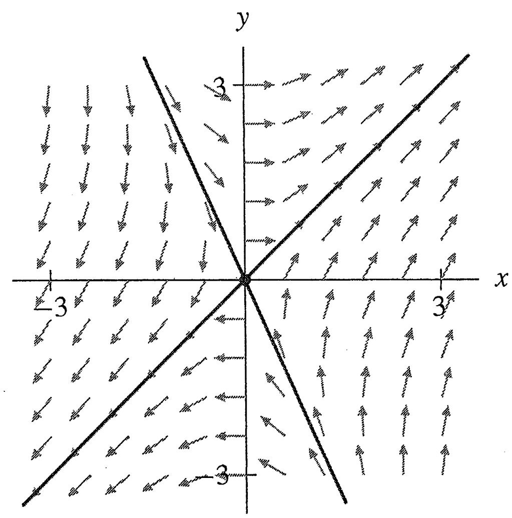
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -4$, we solve the system of equations

$$\begin{cases} 5x_1 + 4y_1 = -4x_1 \\ 9x_1 = -4y_1 \end{cases}$$

and obtain $9x_1 = -4y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2 = 9$ must satisfy the equation $y_2 = x_2$.

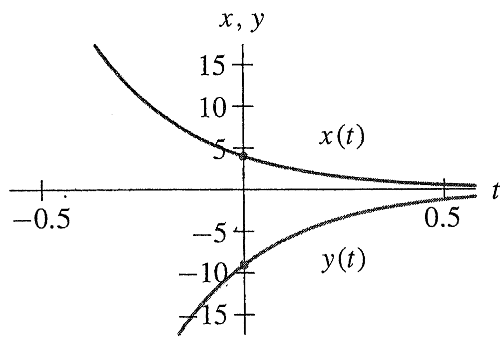
(c)



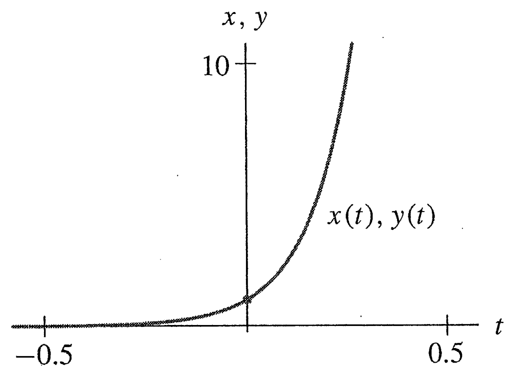
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (4, -9)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (1, 1)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-4t}\begin{pmatrix} 4 \\ -9 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The (identical) $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-4t} \begin{pmatrix} 4 \\ -9 \end{pmatrix} + k_2 e^{9t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

7. **(a)** The characteristic polynomial is

$$(3 - \lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0,$$

and therefore the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

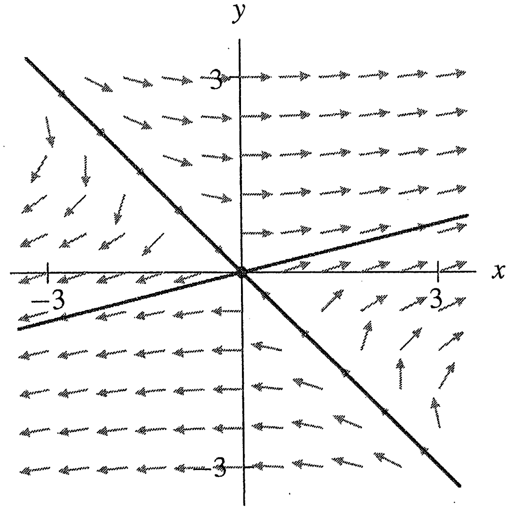
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -1$, we solve the system of equations

$$\begin{cases} 3x_1 + 4y_1 = -x_1 \\ x_1 = -y_1 \end{cases}$$

and obtain $y_1 = -x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2 = 4y_2$ for $\lambda_2 = 4$.

(c)



(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (1, -1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (4, 1)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

11. The eigenvalues are the roots of the characteristic polynomial, so they are solutions of

$$(-2 - \lambda)(1 - \lambda) - 4 = \lambda^2 + \lambda - 6 = 0.$$

Hence, $\lambda_1 = 2$ and $\lambda_2 = -3$ are the eigenvalues.

To find the eigenvectors for the eigenvalue $\lambda_1 = 2$, we solve

$$\begin{cases} -2x_1 - 2y_1 = 2x_1 \\ -2x_1 + y_1 = 2y_1, \end{cases}$$

so $y_1 = -2x_1$ is the line of eigenvectors. In particular, $(1, -2)$ is an eigenvector for $\lambda_1 = 2$.

Similarly, the line of eigenvectors for $\lambda_2 = -3$ is given by $x_1 = 2y_1$. In particular, $(2, 1)$ is an eigenvector for $\lambda_2 = -3$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(a) Given the initial condition $\mathbf{Y}(0) = (1, 0)$, we must solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for k_1 and k_2. This vector equation is equivalent to the two scalar equations

$$\begin{cases} k_1 + 2k_2 = 1 \\ -2k_1 + k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 1/5$ and $k_2 = 2/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{1}{5} e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \tfrac{2}{5} e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(b) Given the initial condition $\mathbf{Y}(0) = (0, 1)$ we must solve

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for k_1 and k_2. This vector equation is equivalent to the two scalar equations

$$\begin{cases} k_1 + 2k_2 = 0 \\ -2k_1 + k_2 = 1. \end{cases}$$

Solving these equations, we obtain $k_1 = -2/5$ and $k_2 = 1/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = -\tfrac{2}{5}e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \tfrac{1}{5}e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(c) The initial condition $\mathbf{Y}(0) = (1, -2)$ is an eigenvector for the eigenvalue $\lambda_1 = 2$. Hence, the solution with this initial condition is

$$\mathbf{Y}(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

12. The characteristic polynomial is

$$(3 - \lambda)(-2 - \lambda) = 0,$$

and therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 3$, we solve the system of equations

$$\begin{cases} 3x_1 = 3x_1 \\ x_1 - 2y_1 = 3y_1 \end{cases}$$

and obtain

$$5y_1 = x_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = 3$ is $\mathbf{V}_1 = (5, 1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (0, 1)$ for $\lambda_2 = -2$.

The general solution to this linear system is therefore

$$\mathbf{Y}(t) = k_1 e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = 1 \\ k_1 + k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 1/5$ and $k_2 = -1/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{1}{5}e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \tfrac{1}{5}e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

15. Given any vector $\mathbf{Y}_0 = (x_0, y_0)$, we have

$$\mathbf{A}\mathbf{Y}_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} ax_0 \\ ay_0 \end{pmatrix} = a \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = a\mathbf{Y}_0.$$

Therefore, every nonzero vector is an eigenvector associated to the eigenvalue a.

16. The characteristic polynomial of $\mathbf{A}$ is

$$(a - \lambda)(d - \lambda) = 0,$$

and thus the eigenvalues of $\mathbf{A}$ are $\lambda_1 = a$ and $\lambda_2 = d$.

To find the eigenvectors $\mathbf{V}_1 = (x_1, y_1)$ associated to $\lambda_1 = a$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_1 = a\mathbf{V}_1$$

for all possible vectors $\mathbf{V}_1$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_1 + by_1 = ax_1 \\ dy_1 = ay_1. \end{cases}$$

Since $a \neq d$, the second equation implies that $y_1 = 0$. If so, then the first equation is satisfied for all x_1. In other words, the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue a are the vectors of the form $(x_1, 0)$.

To find the eigenvectors $\mathbf{V}_2 = (x_2, y_2)$ associated to $\lambda_2 = d$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_2 = d\mathbf{V}_2$$

for all possible vectors $\mathbf{V}_2$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_2 + by_2 = dx_2 \\ dy_2 = dy_2. \end{cases}$$

The second equation always holds, so the eigenvectors $\mathbf{V}_2$ are those vectors that satisfy the equation $ax_2 + by_2 = dx_2$, which can be rewritten as

$$by_2 = (d - a)x_2.$$

These vectors form a line through the origin of slope $(d - a)/b$.

17. The characteristic polynomial of $\mathbf{B}$ is

$$\lambda^2 - (a + d)\lambda + ad - b^2.$$

The roots of this polynomial are

$$\begin{aligned} \lambda &= \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - b^2)}}{2} \\ &= \frac{a + d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2} \\ &= \frac{a + d \pm \sqrt{(a - d)^2 + 4b^2}}{2}. \end{aligned}$$

Since the discriminant $D = (a-d)^2 + 4b^2$ is always nonnegative, the roots λ are real. Therefore, the matrix **B** has real eigenvalues. If $b \neq 0$, then D is positive and hence **B** has two distinct eigenvalues. (The only way to have only one eigenvalue is for $D = 0$).

18. The characteristic equation is

$$(a-\lambda)(-\lambda) - bc = \lambda^2 - a\lambda - bc = 0.$$

Finding the roots via the quadratic formula, we obtain the eigenvalues

$$\frac{a \pm \sqrt{a^2 + 4bc}}{2}.$$

Note that these eigenvalues are very different from the case where the matrix is upper triangular (see Exercise 16). For example, they are not necessarily real numbers because $a^2 + 4bc$ can be negative.

19. **(a)** To form the system, we introduce the new dependent variable $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -p\frac{dy}{dt} - qy = -pv - qy.$$

Written in matrix form this system where $\mathbf{Y} = (y, v)$, we have

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}.$$

(b) The characteristic polynomial is

$$(0-\lambda)(-p-\lambda) + q = \lambda^2 + p\lambda + q.$$

(c) The roots of this polynomial (the eigenvalues) are

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

(d) The roots are distinct real numbers if the discriminant $D = p^2 - 4q$ is positive. In other words, the roots are distinct real numbers if $p^2 > 4q$.

(e) Since q is positive, $p^2 - 4q < p^2$, so we know that $\sqrt{p^2 - 4q} < \sqrt{p^2} = p$. Since the numerator in the expression for the eigenvalues is $-p \pm \sqrt{p^2 - 4q}$, we see that it must be negative. Since the denominator is positive, the eigenvalues must be negative.

20. **(a)** The parameters $m = 1$, $k = 4$, and $b = 5$ yield the second-order equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0.$$

Given $v = dy/dt$, the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -4y - 5v.$$