Multiplying both sides by $\mu(t)$, we obtain

$$(1+t)\frac{dy}{dt} + y = (1+t)t^2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d((1+t)y)}{dt} = t^3 + t^2,$$

and integrating both sides with respect to t, we obtain

$$(1+t)y = \frac{t^4}{4} + \frac{t^3}{3} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{3t^4 + 4t^3 + 12c}{12(t+1)}.$$

4. We rewrite the equation in the form

$$\frac{dy}{dt} + 2ty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 4.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{t^2}y)}{dt} = 4,$$

and integrating both sides with respect to t, we obtain

$$e^{t^2}y = 4t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

5. Note that the integrating factor is

$$\mu(t) = e^{\int(-2t/(1+t^2))\,dt} = e^{-\ln(1+t^2)} = \left(e^{\ln(1+t^2)}\right)^{-1} = \frac{1}{1+t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{1+t^2}\frac{dy}{dt} - \frac{2t}{(1+t^2)^2}y = \frac{3}{1+t^2}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{1+t^2}\right) = \frac{3}{1+t^2}.$$

Integrating both sides with respect to t, we obtain

$$\frac{y}{1+t^2} = 3\arctan(t) + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = (1+t^2)(3\arctan(t) + c).$$

6. Note that the integrating factor is

$$\mu(t) = e^{\int(-2/t)\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = t^{-2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t^{-2}\frac{dy}{dt} - 2t^{-3}y = te^t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(t^{-2}y)}{dt} = te^t,$$

and integrating both sides with respect to t, we obtain

$$t^{-2}y = (t-1)e^t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t^2(t-1)e^t + ct^2.$$

Integrating both sides with respect to t, we obtain

$$\frac{y}{t+1} = 2t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = (2t^2 + c)(t + 1) = 2t^3 + 2t^2 + ct + c.$$

To find the solution that satisfies the initial condition $y(1) = 10$, we evaluate the general solution at $t = 1$ and obtain $c = 3$. The desired solution is

$$y(t) = 2t^3 + 2t^2 + 3t + 3.$$

9. In Exercise 1, we derived the general solution

$$y(t) = t + \frac{c}{t}.$$

To find the solution that satisfies the initial condition $y(1) = 3$, we evaluate the general solution at $t = 1$ and obtain $c = 2$. The desired solution is

$$y(t) = t + \frac{2}{t}.$$

10. In Exercise 4, we derived the general solution

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

To find the solution that satisfies the initial condition $y(0) = 3$, we evaluate the general solution at $t = 0$ and obtain $c = 3$. The desired solution is

$$y(t) = 4te^{-t^2} + 3e^{-t^2}.$$

11. Note that the integrating factor is

$$\mu(t) = e^{\int -(2/t)\,dt} = e^{-2\int (1/t)\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = \frac{1}{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{t^2}\frac{dy}{dt} - \frac{2y}{t^3} = 2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = 2,$$

and integrating both sides with respect to t, we obtain

$$\frac{y}{t^2} = 2t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 2t^3 + ct^2.$$

To find the solution that satisfies the initial condition $y(-2) = 4$, we evaluate the general solution at $t = -2$ and obtain

$$-16 + 4c = 4.$$

Hence, $c = 5$, and the desired solution is

$$y(t) = 2t^3 + 5t^2.$$

12. Note that the integrating factor is

$$\mu(t) = e^{\int(-3/t)\,dt} = e^{-3\ln t} = e^{\ln(t^{-3})} = t^{-3}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t^{-3}\frac{dy}{dt} - 3t^{-4}y = 2e^{2t}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(t^{-3}y)}{dt} = 2e^{2t},$$

and integrating both sides with respect to t, we obtain

$$t^{-3}y = e^{2t} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t^3(e^{2t} + c).$$

To find the solution that satisfies the initial condition $y(1) = 0$, we evaluate the general solution at $t = 1$ and obtain $c = -e^2$. The desired solution is

$$y(t) = t^3(e^{2t} - e^2).$$

13. We rewrite the equation in the form

$$\frac{dy}{dt} - (\sin t)y = 4$$

and note that the integrating factor is

$$\mu(t) = e^{\int(-\sin t)\,dt} = e^{\cos t}.$$

and integrating both sides with respect to t, we obtain

$$\left(e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\right) y = \int t \left(e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\right) dt.$$

These integrals are also impossible to express in terms of elementary functions, so we write the general solution in the form

$$y(t) = \left(e^{\int \frac{1}{\sqrt{t^3-3}}\,dt}\right) \int t \left(e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\right) dt.$$

19. We rewrite the equation in the form

$$\frac{dy}{dt} - aty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int(-at)\,dt} = e^{-at^2/2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{-at^2/2}\frac{dy}{dt} - ate^{-at^2/2}y = 4e^{-t^2}e^{-at^2/2}.$$

Applying the Product Rule to the left-hand side and simplifying the right-hand side, we see that this equation is the same as

$$\frac{d(e^{-at^2/2}y)}{dt} = 4e^{-(1+a/2)t^2}.$$

Integrating both sides with respect to t, we obtain

$$e^{-at^2/2}y = \int 4e^{-(1+a/2)t^2}\,dt.$$

The integral on the right-hand side can be expressed in terms of elementary functions only if $1 + a/2 = 0$ (that is, if the factor involving e^{t^2} really isn't there). Hence, the only value of a that yields an integral we can express in terms of elementary functions form is $a = -2$ (see Exercise 4).

20. We rewrite the equation in the form

$$\frac{dy}{dt} - t^r y = 4$$

and note that the integrating factor is

$$\mu(t) = e^{-\int t^r\,dt}.$$

There are two cases to consider.

(a) If $r \neq -1$, then

$$\mu(t) = e^{-t^{r+1}/(r+1)}.$$

Multiplying both sides of the differential equation by $\mu(t)$, we obtain

$$\left(e^{-t^{r+1}/(r+1)}\right)\frac{dy}{dt} - t^r\left(e^{-t^{r+1}/(r+1)}\right)y = 4\left(e^{-t^{r+1}/(r+1)}\right).$$