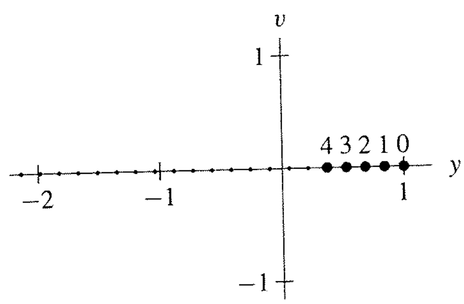
10. The natural frequency and the forcing frequency are the same, $1/\pi$. The solution will have a resonance term of the form $t \sin 2t$ and/or $t \cos 2t$. Except for the resonance term(s), all the other terms in the solution are periodic with period π; so, the Poincaré return map does not see the non-resonance terms. For every time increase of π the amplitude of the resonance term(s) increases linearly, thus one expects the Poincaré return map to be a sequence of equidistant points along a straight line.



REVIEW EXERCISES FOR CHAPTER 5

1. Since the equilibrium point is at the origin and the system has only polynomial terms, the linearized system is just the linear terms in dx/dt and dy/dt, that is,

$$\frac{dx}{dt} = x$$
$$\frac{dy}{dt} = -2y.$$

2. From the linearized system in Exercise 1, we see (without any calculation) that the eigenvalues are 1 and -2. Hence, the origin is a saddle.

3. The Jacobian matrix for this system is

$$\begin{pmatrix} 2x + 3\cos 3x & 0 \\ -y\cos xy & 2 - x\cos xy \end{pmatrix},$$

and evaluating at $(0, 0)$, we get

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

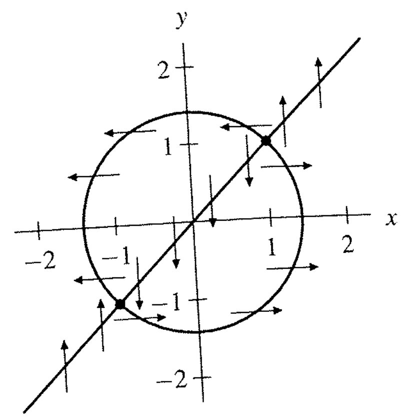
So the linearized system at the origin is

$$\frac{dx}{dt} = 3x$$
$$\frac{dy}{dt} = 2y.$$

4. From the linearized system in Exercise 3, we see (without any calculation) that the eigenvalues are 3 and 2. Hence, the origin is a source.

5. The x-nullcline is where $dx/dt = 0$, that is, the line $y = x$. The y-nullcline is where $dy/dt = 0$, that is, the circle $x^2 + y^2 = 2$.

Along the x-nullcline, $dy/dt < 0$ if and only if $-\sqrt{2} < x < \sqrt{2}$. Along the y-nullcline, $dx/dt < 0$ if and only if $y > x$.



6. This system is not a Hamiltonian system. If it were, then we would have

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} \quad \text{and} \quad -\frac{\partial H}{\partial x} = \frac{dy}{dt}$$

for some function $H(x, y)$. In that case, equality of mixed partials would imply that

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) = -\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right).$$

For this system, we have

$$\frac{\partial}{\partial x}\left(\frac{dx}{dt}\right) = 2y \quad \text{and} \quad -\frac{\partial}{\partial y}\left(\frac{dy}{dt}\right) = -2y.$$

Since these two partials do not agree, no such function $H(x, y)$ exists.

7. This system is not a gradient system. If it were, then we would have

$$\frac{\partial G}{\partial x} = \frac{dx}{dt} \quad \text{and} \quad \frac{\partial G}{\partial y} = \frac{dy}{dt}$$

for some function $G(x, y)$. In that case, equality of mixed partials would imply that

$$\frac{\partial}{\partial y}\left(\frac{dx}{dt}\right) = \frac{\partial}{\partial x}\left(\frac{dy}{dt}\right).$$

For this system, we have

$$\frac{\partial}{\partial y}\left(\frac{dx}{dt}\right) = 2x + 2y \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{dy}{dt}\right) = 2x.$$

Since these two partials do not agree, no such function $G(x, y)$ exists.

8. Some possibilities are:

- The solution is unbounded. That is, either $|x(t)| \to \infty$ or $|y(t)| \to \infty$ (or both) as t increases.
- Similarly, $x(t)$ or $y(t)$ (or both) oscillate with increasing amplitude as t increases (similar to $t \sin t$).
- The solution tends to an equilibrium point.
- The solution tends to a periodic solution, as in the Van der Pol equation (see Section 5.1).
- The solution tends to a curve consisting of equilibrium points and solutions connecting equilibrium points.

9. If the system is a linear system, then all nonequilibrium solutions tend to infinity as t increases, that is, $|\mathbf{Y}(t)| \to \infty$ as $t \to \infty$.

If the system is not linear, it is possible for a solution to spiral toward a periodic solution. For example, consider the Van der Pol equation discussed in Section 5.1. (These two behaviors are the only possibilities.)

10. Since a solution that enters the first quadrant cannot leave, the origin cannot be a spiral sink, a spiral source, or a center.

However, a sink, a saddle, or a source are all possibilities. For example,

$$\frac{dx}{dt} = -2x + y$$
$$\frac{dy}{dt} = x - y$$

has a sink at the origin,

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x$$

has a saddle at the origin, and

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = x + y$$

has a source at the origin.

11. True. The x-nullcline is where $dx/dt = 0$ and the y-nullcline is where $dy/dt = 0$, so any point in common must be an equilibrium point.

12. False. For example, both nullclines for the system

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = y - x$$

are the line $y = x$. Moreover, since the nullclines are identical, all points on the line are equilibrium points.

13. False. These two numbers are the diagonal entries of the Jacobian matrix. The other two entries of the Jacobian matrix also affect the eigenvalues.

14. False. The Jacobian matrix at an equilibrium point (x_0, y_0) is

$$\begin{pmatrix} f'(x_0) & 0 \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix},$$

so its eigenvalues are $f'(x_0)$ and

$$\frac{\partial g}{\partial y}(x_0, y_0).$$

Since this partial derivative could be positive, negative, or zero, the equilibrium point could be a source, a saddle, or one of the zero eigenvalue types.

15. **(a)** Setting $dx/dt = 0$ and $dy/dt = 0$, we obtain the simultaneous equations

$$\begin{cases} x - 3y^2 = 0 \\ x - 3y - 6 = 0. \end{cases}$$

Solving for x and y yields the equilibrium points $(12, 2)$ and $(3, -1)$.

To determine the type of an equilibrium point, we compute the Jacobian matrix. We get

$$\begin{pmatrix} 1 & -6y \\ 1 & -3 \end{pmatrix}.$$

At $(12, 2)$, the Jacobian is

$$\begin{pmatrix} 1 & -12 \\ 1 & -3 \end{pmatrix},$$

and its eigenvalues are $-1 \pm 2\sqrt{2}\,i$. Hence, $(12, 2)$ is a spriral sink.

At $(3, -1)$, the Jacobian matrix is

$$\begin{pmatrix} 1 & 6 \\ 1 & -3 \end{pmatrix},$$

and the eigenvalues are $-1 \pm \sqrt{10}$. So $(3, -1)$ is a saddle.

25. **(a)** The equilibrium points are the solutions of

$$\begin{cases} y^2 - x^2 - 1 = 0 \\ 2xy = 0, \end{cases}$$

that is, $(0, \pm 1)$.

The Jacobian matrix is

$$\begin{pmatrix} -2x & 2y \\ 2y & 2x \end{pmatrix}.$$

At $(0, 1)$, the Jacobian is

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^2 - 4$, so its eigenvalues are $\lambda = \pm 2$. The equilibrium point is a saddle.

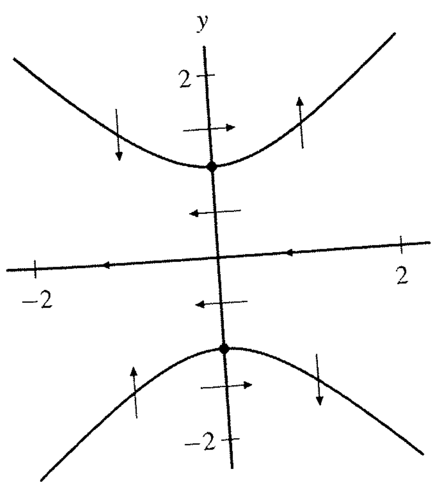
At $(0, -1)$, the Jacobian is

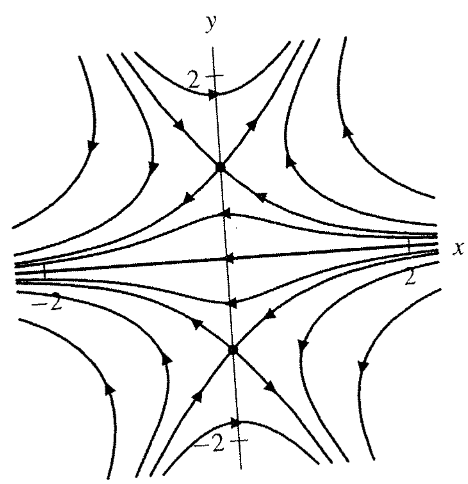
$$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^2 - 4$, so its eigenvalues are $\lambda = \pm 2$. The equilibrium point is a saddle.

(b) The x-nullcline is the hyperbola $y^2 - x^2 = 1$, and the y-nullclines are the x- and y-axes.

In the following figures, the nullclines are on the left and the phase portrait is on the right.





(c) To see if the system is Hamiltonian, we compute

$$\frac{\partial (y^2 - x^2 - 1)}{\partial x} = -2x \quad \text{and} \quad -\frac{\partial (2xy)}{\partial y} = -2x.$$

Since these partials agree, the system is Hamiltonian.

The Hamiltonian is a function $H(x, y)$ such that

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} = y^2 - x^2 - 1 \quad \text{and} \quad \frac{\partial H}{\partial x} = -\frac{dy}{dt} = -2xy.$$

We integrate the second equation with respect to x to see that

$$H(x, y) = -x^2 y + \phi(y),$$

where $\phi(y)$ represents the terms whose derivative with respect to x are zero. Using this expression for $H(x, y)$ in the first equation, we obtain

$$-x^2 + \phi'(y) = y^2 - x^2 - 1.$$

Hence, $\phi'(y) = y^2 - 1$, and we can take $\phi(y) = \frac{1}{3}y^3 - y$. The function

$$H(x, y) = -x^2 y + \frac{y^3}{3} - y$$

is a Hamiltonian function for this system.

(d) To see if the system is a gradient system, we compute

$$\frac{\partial(y^2 - x^2 - 1)}{\partial y} = 2y \quad \text{and} \quad \frac{\partial(2xy)}{\partial x} = 2y.$$

Since these partials agree, the system is a gradient system.

We must now find a function $G(x, y)$ such that

$$\frac{\partial G}{\partial x} = \frac{dx}{dt} = y^2 - x^2 - 1 \quad \text{and} \quad \frac{\partial G}{\partial y} = \frac{dy}{dt} = 2xy.$$

Integrating the second equation with respect to y, we obtain

$$G(x, y) = xy^2 + h(x),$$

where $h(x)$ represents the terms whose derivative with respect to y are zero.

Using this expression for $G(x, y)$ in the first equation, we obtain

$$y^2 + h'(x) = y^2 - x^2 - 1.$$

Hence, $h'(x) = -x^2 - 1$, and we can take $h(x) = -\frac{1}{3}x^3 - x$. The function

$$G(x, y) = xy^2 - \frac{x^3}{3} - x$$

is the required function.

26. **(a)** Letting $y = dx/dt$, we obtain the system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = 3x - x^3 - 2y.$$

From the first equation, we see that $y = 0$ for any equilibrium point. Substituting $y = 0$ in the equation $3x - x^3 - 2y = 0$ yields $x = 0$ or $x^2 = 3$. Hence, the equilibria are $(0, 0)$ and $(\pm\sqrt{3}, 0)$.

(b) The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ 3 - 3x^2 & -2 \end{pmatrix}.$$

Evaluating the Jacobian at (0, 0) yields

$$\begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix},$$

which has eigenvalues -3 and 1. Hence, the origin is a saddle. At $(\pm\sqrt{3}, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -6 & -2 \end{pmatrix},$$

which has eigenvalues $-1 \pm i\sqrt{5}$. Hence, these two equilibria are spiral sinks.

27. To see if the system is Hamiltonian, we compute

$$\frac{\partial(-3x + 10y)}{\partial x} = -3 \quad \text{and} \quad -\frac{\partial(-x + 3y)}{\partial y} = -3.$$

Since these partials agree, the system is Hamiltonian.

To find the Hamiltonian function, we use the fact that

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} = -3x + 10y.$$

Integrating with respect to y gives

$$H(x, y) = -3xy + 5y^2 + \phi(x),$$

where $\phi(x)$ represents the terms whose derivative with respect to y are zero. Differentiating this expression for $H(x, y)$ with respect to x gives

$$-3y + \phi'(x) = -\frac{dy}{dt} = x - 3y.$$

We choose $\phi(x) = \frac{1}{2}x^2$ and obtain the Hamiltonian function

$$H(x, y) = -3xy + 5y^2 + \frac{x^2}{2}.$$

We know that the solution curves of a Hamiltonian system remain on the level sets of the Hamiltonian function. Hence, solutions of this system satisfy the equation

$$-3xy + 5y^2 + \frac{x^2}{2} = h$$

for some constant h. Multiplying through by 2 yields the equation

$$x^2 - 6xy + 10y^2 = k$$

where $k = 2h$ is a constant.

28. **(a)** To see if the system is Hamiltonian, we compute

$$\frac{\partial(ax+by)}{\partial x} = a \quad \text{and} \quad -\frac{\partial(cx+dy)}{\partial y} = -d.$$

For these partials to agree, we must have $a = -d$.

Assuming that $d = -a$, we want a function $H(x, y)$ such that

$$\frac{\partial H}{\partial y} = \frac{dx}{dt} = ax + by \quad \text{and} \quad \frac{\partial H}{\partial x} = -\frac{dy}{dt} = -cx + ay.$$

We integrate the second equation with respect to x to see that

$$H(x, y) = -\frac{c}{2}x^2 + axy + \phi(y),$$

where $\phi(y)$ represents the terms whose derivative with respect to x are zero.

Using this expression for $H(x, y)$ in the first equation, we obtain

$$ax + \phi'(y) = ax + by.$$

In other words, $\phi'(y) = by$, and we can take $\phi(y) = by^2/2$. The function

$$H(x, y) = -\frac{c}{2}x^2 + axy + \frac{b}{2}y^2$$

is a Hamiltonian function for this system if $d = -a$.

(b) To see if the system is a gradient system, we compute

$$\frac{\partial(ax+by)}{\partial y} = b \quad \text{and} \quad \frac{\partial(cx+dy)}{\partial x} = c.$$

The linear system is a gradient system if $b = c$.

Assuming that $b = c$, we want a function $G(x, y)$ such that

$$\frac{\partial G}{\partial x} = \frac{dx}{dt} = ax + by \quad \text{and} \quad \frac{\partial G}{\partial y} = \frac{dy}{dt} = bx + dy.$$

Integrating the first equation with respect to x, we obtain

$$G(x, y) = \frac{a}{2}x^2 + bxy + h(y),$$

where $h(y)$ represents the terms whose derivative with respect to y are zero.

Using this expression for $G(x, y)$ in the second equation, we obtain

$$bx + h'(y) = bx + dy.$$

Hence, $h'(y) = dy$, and we can take $h(y) = dy^2/2$. The function

$$G(x, y) = \frac{a}{2}x^2 + bxy + \frac{d}{2}y^2$$

is the required function if $c = b$.

(c) The system is Hamiltonian if $d = -a$ and gradient if $b = c$. Both conditions are satisfied if the system has the form

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mathbf{Y}.$$

The eigenvalues of the coefficient matrix are $\pm\sqrt{a^2 + b^2}$, so the origin is a saddle if the system is both Hamiltonian and gradient.

(d) Any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $d \neq -a$ and $b \neq c$ gives a system that is neither Hamiltonian nor gradient. (Recall that both gradient and Hamiltonian systems cannot have equilibrium points that are spiral sources or spiral sinks.)

29. **(a)** Since θ represents an angle in this model, we restrict θ to the interval $-\pi < \theta < \pi$.

The equilibria must satisfy the equations

$$\begin{cases} \cos\theta = s^2 \\ \sin\theta = -s^2. \end{cases}$$

Therefore,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-s^2}{s^2} = -1,$$

and consequently, $\theta = -\arctan 1 = -\pi/4$.

To find s, we note that $s^2 = \cos(-\pi/4) = 1/\sqrt{2}$. Hence, $s = 1/\sqrt[4]{2}$, and the only equilibrium point is

$$(\theta, s) = \left(-\frac{\pi}{4}, \frac{1}{\sqrt[4]{2}}\right).$$

(b) The Jacobian matrix for this system is

$$\begin{pmatrix} \dfrac{\sin\theta}{s} & 1 + \dfrac{\cos\theta}{s^2} \\ -\cos\theta & -2s \end{pmatrix}.$$

Evaluating at the equilibrium point, we get

$$\begin{pmatrix} -2^{-3/4} & 2 \\ -2^{-1/2} & -2^{3/4} \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^2 + \frac{1 + 2\sqrt{2}}{2^{3/4}}\lambda + (1 + \sqrt{2}).$$

Since

$$\left(\frac{1 + 2\sqrt{2}}{2^{3/4}}\right)^2 - 4(1 + \sqrt{2}) < 0,$$