*Math 341–Fall 2006*

Midterm Exam 2–November 20th, 2006

Name: Buckmire

*You may use a calculator, your note card and something to write with. You must attach your notecard to the exam when you turn it in. You cannot use any differential equation solving or integrating capacities that are on your calculator.*

All work must be shown to receive full credit for any answer.

**1.** *(10 points)* Find the general solution to the second-order equation

$$\frac{d^2y}{dt^2} + 4y = 3\cos t.$$

$$L = \frac{d^2}{dt^2} + 4$$

$$Ly = 3\cos(t)$$

$$Ly_h = 0$$

$$y(t) = A\cos(2t) + B\sin(2t)$$

$$Ly_p = 3\cos t$$

$$y_p = K\cos(t)$$

$$y_p' = -K\sin(t)$$

$$y_p'' = -K\cos(t)$$

$$-K\cos(t) + 4K\cos(t) = 3\cos(t)$$

$$3K\cos(t) = 3\cos(t)$$

$$K = 1$$

$$y(t) = A\cos(2t) + B\sin(2t) + \cos(t)$$

**3.** *(10 points)* **Multiple Choice.** Give reasons or show work.

*(a)* Consider the matrix

$$B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}.$$

Which of the following are eigenvalues for $B$?

$A.\ 0$; $B.\ 1$; $C.\ i\sqrt{2}$; $D.\ -2$; $E.\ \sqrt{2}$; $F.\ 2$; $G.$ None of these.

eigenvalues are 1

*(b)* Each of the following matrices has an eigenvalue equal to 2:

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 6 \end{pmatrix} B = \begin{pmatrix} 6 & 3 \\ 0 & 2 \end{pmatrix} C = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}$$

Here are four possible eigenvectors associated to the eigenvalue 2. Match the letter of the matrix with the number of the vector that could serve as an eigenvector:

$$1.\ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad 2.\ v_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \quad 3.\ v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad 4.\ v_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A \leftrightarrow 1\ \vec{v}_1$$

$$\begin{pmatrix} 6 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix} = 2\begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$$B \leftrightarrow 2\ \vec{v}_2$$

$$\begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$C \leftrightarrow 3\ \vec{v}_3$$

**4.** *(10 points)* Consider the following system of differential equations:

$$\frac{dx}{dt} = y - x^2,$$
$$\frac{dy}{dt} = y^2 - x.$$

Classify all equilibrium points for this system.

$0 = y - x^2 \Rightarrow y = x^2$

$0 = y^2 - x$

$(x^2)^2 - x = 0$

$x^4 - x = 0$

$(x^3 - 1)x = 0$

$(x^2 + x + 1)(x - 1)x = 0$

$x = 0, 1$

$(0,0)$ and $(1,1)$

$J = \begin{pmatrix} -2x & 1 \\ -1 & 2y \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$ $\quad (0,0)$ is a center

$J(1,1) = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$ $\quad \lambda^2 - 3 = 0 \Rightarrow \lambda = \pm\sqrt{3}$ $\quad (1,1)$ is a saddle

**Do only one of the next two problems (5 and 6). Make an "X" over the problem you do not want graded. If there is work on both and no cross out, only the first problem will be graded.**

5. *(10 points)* Consider the nonlinear system of differential equations:

$$\frac{dx}{dt} = x(-2x - y + 40),$$
$$\frac{dy}{dt} = y(-x - y + 30).$$

*1.* Find all equilibrium points for this system.

*2.* Draw the nullclines for this system and label each nullclines as either "x-nullcline" or "y-nullcline."

*3.* Sketch arrows in the phase plane indicating the direction of motion along the nullclines as well as in the regions between the nullclines.

*4.* Sketch representative solution curves in the phase plane for this system.

$X = 0$ or $-2x - y + 40 = 0 \Leftrightarrow y = 40 - 2x$

AND

$y = 0$ or $-x - y + 30 = 0 \Leftrightarrow y = 30 - x$

Equilibrium at $x = 0, y = 30$

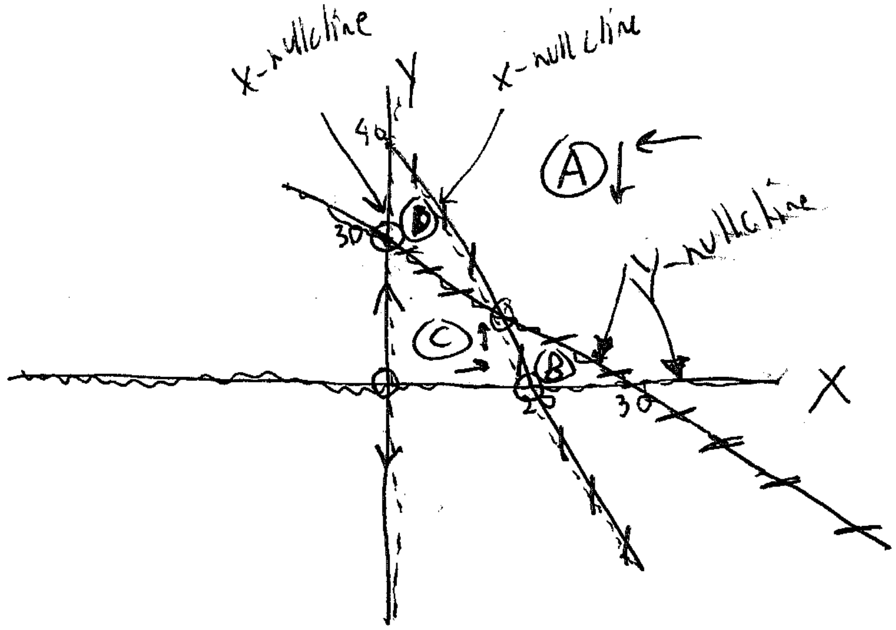
$x = 0, y = 0$

$x = 20, y = 0$

$40 - 2x = 30 - x$

$10 = x$

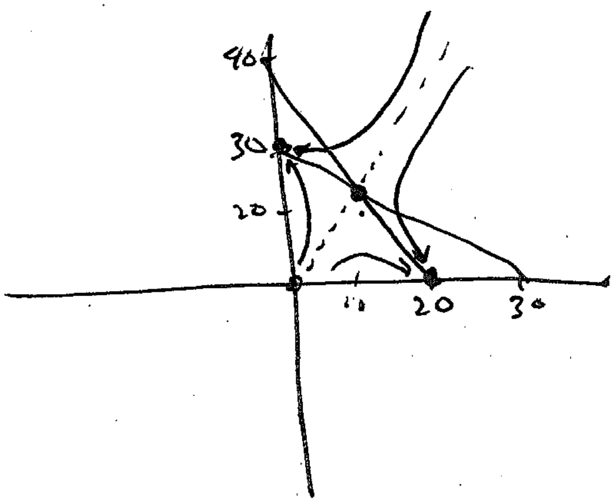
$20 = y$



EST Working space continued for problem (4).

| | | | | |
|---|---|---|---|---|
| (40,1) | (A) | $\frac{dx}{dt} < 0, \frac{dy}{dt} < 0$ | DOWN - LEFT | $\downarrow \leftarrow$ |
| (1,1) | (E) | $\frac{dx}{dt} > 0, \frac{dy}{dt} > 0$ | UP-RIGHT | $\uparrow \rightarrow$ |
| (25,1) | (B) | $\frac{dx}{dt} < 0, \frac{dy}{dt} > 0$ | UP- LEFT | $\uparrow \leftarrow$ |
| | ~~(B)~~ | | DOWN -RIGHT | $\downarrow \rightarrow$ |
| (2,30) | (B) | $\frac{dx}{dt} > 0, \frac{dy}{dt} < 0$ | | |

Pretty ugly sketch



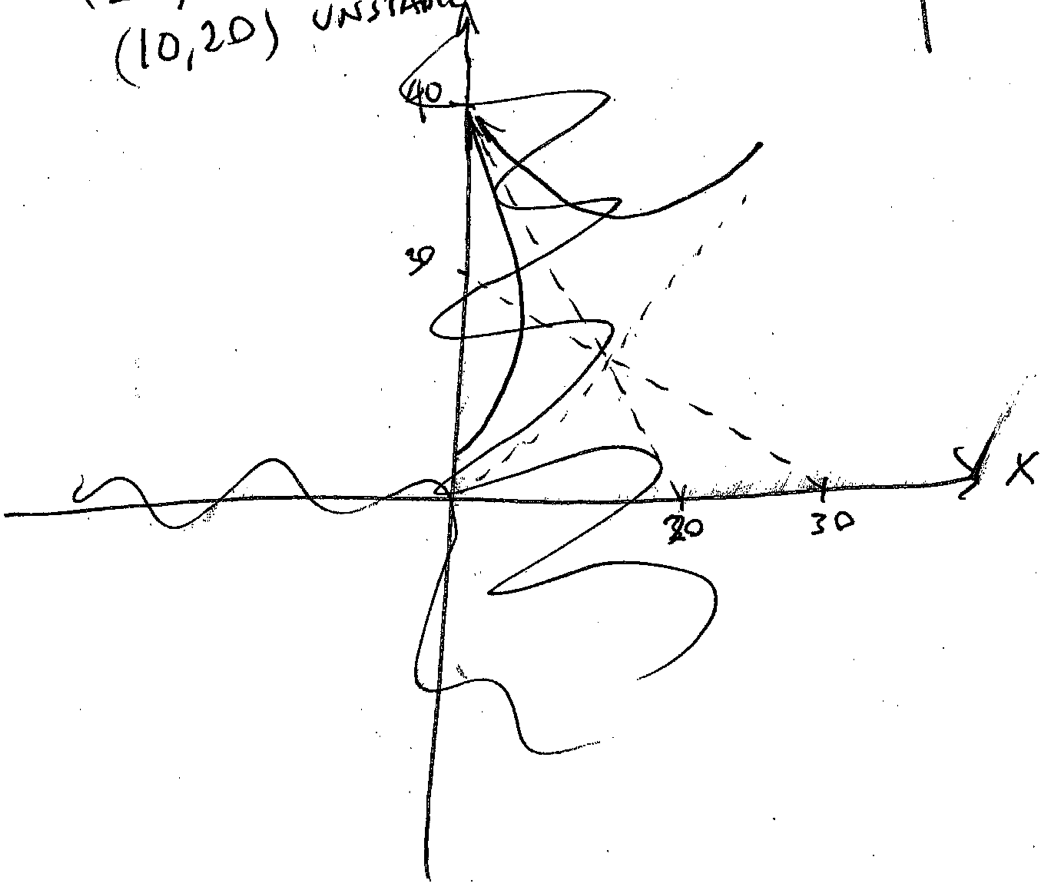

Equilibria at

(0,0) UNSTABLE

(0,30) STABLE

(20,0) STABLE

(10,20) UNSTABLE



**6.** *(10 points)* Consider the following one-parameter family of linear systems:

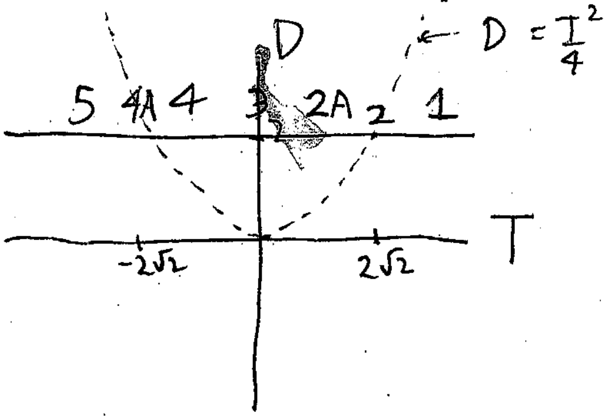
$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & -1 \\ 2 & 0 \end{pmatrix} \mathbf{Y}.$$

*1.* Find the trace and determinant for these systems.

$T = a + 0 = a$

$D = 2 = a \cdot 0 - (2)(-1)$

*2.* Draw the curve in the trace-determinant plane that corresponds to this system.




4A

$\begin{pmatrix} -2\sqrt{2} & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda^2 + 2\sqrt{2}\lambda + 2 = 0$

$(\lambda + \sqrt{2})^2 = 0$

$\lambda = -\sqrt{2}, -\sqrt{2}$

IMPROPER NODE (SINK)

5

$\begin{pmatrix} -3 & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda^2 + 3\lambda + 2$

$\lambda = -1, -2$

SINK

*3.* On your trace-determinant plane, indicate with a number each region where the phase portrait changes.

*4.* In the space below, write the number for each different type of phase portrait, and characterize the origin for that phase portrait. Distinguish between star nodes and degenerate nodes if necessary.

1

$\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 1, 2$

SOURCE

2

$\begin{pmatrix} 2\sqrt{2} & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda^2 - 2\sqrt{2}\lambda + 2 = 0$

$(\lambda - \sqrt{2})^2 = 0$

$\lambda = \sqrt{2}, \sqrt{2}$

IMPROPER NODE (SOURCE)

3A ~~4~~

$\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \frac{2 \pm \sqrt{-4}}{2}$

$= 1 \pm i$

SPIRAL SOURCE

3 ~~5~~

$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda^2 + 2 = 0$

$\lambda = \pm i\sqrt{2}$

CENTER

4

$\begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}$

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda = \frac{-2 \pm \sqrt{-4}}{2}$

$= -1 \pm i$

SPIRAL SINK