Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 2

 `http://faculty.oxy.edu/ron/math/341`

Class 29: Friday April 15

TITLE *Power Series Solutions of Differential Equations*
CURRENT READING Zill 6.1

Homework Set #12
Zill, Section 6.1: 10*,13*,15*,26*,29*
Zill, Section 6.2: 3*,4*,12*,13* *EXTRA CREDIT 33*
Zill, Section 6.3: 5*,10*,25*,30* *EXTRA CREDIT 33*
Zill, Chapter 6 Review: 4*,5*,10*,15*,20* *EXTRA CREDIT 22*

SUMMARY
An introduction to the very powerful technique of using infinite series to represent the solution to a differential equation. We'll review concepts involving power series.

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1. Review of Power Series

DEFINITION: **Power Series**

An infinite series of the form
$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$
is called a **power series centered around** a.

DEFINITION: **Partial Sum**

A **partial sum**, $S_N(x)$ of an infinite series is the N^{th} degree polynomial formed from a sum of the first $N+1$ terms of the series, i.e. $\displaystyle\sum_{k=0}^{N} c_k(x-a)^k$.

DEFINITION: **Convergence**

A power series is said to be **convergent** at a specified value of x if the sequence of partial sums converges to a finite value, in other words the $\lim_{N\to\infty} S_N(x)$ exists. If the series does not converge at x, then it is said to be **divergent** at x.

DEFINITION: **Radius of Convergence**

A power series is said to have a Radius of Convergence $R > 0$ if the series $\displaystyle\sum_{k=0}^{\infty} c_k(x-a)^k$ converges for all x values $|x-a| < R$ (it will also diverge for $|x-a| > R$.) The open interval $(a-R, a+R)$ is called the **interval of convergence**. The endpoints of the interval at $x = a-R$ and $x = a+R$ may or may not be convergent points. If the series converges for every value of x the radius of convergence is said to be infinite.

DEFINITION: **Absolute Convergence**

In the interval of convergence a power series converges absolutely. In other words,
$\displaystyle\sum_{k=0}^{\infty} |c_k(x-a)^k|$ converges for $|x-a| < R$. If a series converges absolutely, then it converges. The converse is not necessarily true.

DEFINITION: **Power Series as a Function**

A function $f(x)$ is said to be **analytic** at a point a if it can be represented by a power series which includes a in its interval of converence. One can think of a power series with a non-zero radius of convergence as a continuous, differentiable and integrable function where the interval of convergence is its domain of definition. The derivative and anti-derivative of a power series all have the same radius of convergence of the initial power series. Often power series are computed as Taylor Series (or Maclaurin Series when $a = 0$) where $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(x-a)^k$. Typical elementary functions like e^x, $\sin(x)$ and $\cos(x)$ all have power series with an infinite radius of convergence. The function $\ln(x+1)$ is analytic on the interval $[-1, 1)$. It's derivative $\frac{1}{x+1}$ is analytic on the interval $(-1, 1)$.

2. Power Series Solutions of DEs

DEFINITION: **Singular and Ordinary Points**

Consider the general form of the second-order linear differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. When put into standard form $y'' + P(x)y' + Q(x)y = 0$ where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$ the point x_0 is said to be a **ordinary point** of the differential equation if $P(x)$ and $Q(x)$ are analytic at x_0. Otherwise x_0 is called a **singular point** of the differential equation.

THEOREM: **Power Series Solution**

If $x = x_0$ is an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$ we can find two linearly independent solutions which can each be written as power series centered at x_0, i.e. $\sum_{k=0}^{\infty} c_k(x-x_0)^k$ which converge in some interval $|x - x_0| < R$ where R is the distance to the nearest singular point.

EXAMPLE Let's solve $y'' - 2xy' - 2y = 0$ using power series.

Exercise **Zill, page 245, Example 4**. Solve $(x^2+1)y''+xy'-y=0$ and give the interval of convergence of the solution $y(x)$.

ANSWER: $y_1(x) = x$ and $y_2(x) = (1+x^2)^{1/2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \ldots$ where the interval of convergence is $|x| < 1$.

3. Classifying Singular Points

DEFINITION: **Regular and Irregular Singular Points**

A point x_0 is said to be a **regular singular point** of the DE $y'' + P(x)y' + Q(x)y = 0$ if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are BOTH analytic at x_0. A singular point which is not regular is said to be an **irregular singular point**.

EXAMPLE Classify the singular point(s) of the differential equation $(x^2 - 4)^2y'' + 3(x - 2)y' + 5y = 0$.

Exercise Classify the singular point(s) of $x^3y'' - 2xy' + 8y = 0$