Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 2

 http://faculty.oxy.edu/ron/math/341

Class 28: Friday April 8

TITLE *The Dirac Delta Function*
CURRENT READING Zill 7.5

Homework Set #11
Zill, Section 7.3: 3*, 7*, 15*, 22*, 39*, 43* *EXTRA CREDIT 49-54*
Zill, Section 7.4: 1*, 2*, 19*, 27*,33*, 41* *EXTRA CREDIT 45, 49*
Zill, Section 7.5: 3*, 9*
Zill, Chapter 7 Review: 25*,26*,27*,28*,29* *EXTRA CREDIT 37*

SUMMARY
An introduction to the wild and wacky Dirac delta "function."

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1. The Unit Impulse Function

Consider the unit impulse function $\delta_a(t) = \begin{cases} 0, & 0 \leq t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a < t < t_0 + a \\ 0, & t_0 + a < t \end{cases}$

DEFINITION: **Dirac Delta Function** The **Dirac Delta Function** is denoted by $\delta(t - t_0)$ and is the object (it's not really a function) which results when one takes the limit as $a \to 0$ of the unit impulse function $\delta_a(t - t_0)$. In other words, $\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$.

The Dirac Delta Function also has the property that $\int_{-\infty}^{\infty} \delta(t - t_0)\ dt = 1$

THEOREM: **The Laplace Transform of the Dirac Delta Function**
For $t_0 > 0$, $\mathcal{L}[\delta(t - t_0)] = e^{-st_0}$ and $\mathcal{L}^{-1}[e^{-st_0}] = \delta(t - t_0)$. (For more details, see Zill, p. 316).

Interestingly, we can relate the Heaviside function $\mathcal{H}(t)$ and Dirac Delta Function $\delta(t)$. Consider the following integrally defined function $f(x) = \int_{-\infty}^{x} \delta(t - t_0)\ dt$.

Q: What does $f(x)$ look like? **A:** Depends on the relationship between x and t_0. How?

The integral of the ______________________ is the ______________________, and the ______________________ of Heaviside Function is equal to the Dirac Delta Function. (Pretty cool, eh?) Sketch the Heaviside Function and Dirac Delta Function for all t values.

2. Delta Function as Source Term

What's interesting about the Dirac Delta Function is that it allows us to model situations where an instantaneous impulse is applied to a system at a certain time. Laplace Transforms are really the only technique which allow solution of such initial value problems.

EXAMPLE **Zill, page 316, Example 1**. Solve $y'' + y = 4\delta(t - 2\pi)$ where **(a)** $y(0) = 1, \quad y'(0) = 0$ and **(b)** $y(0) = 0, \quad y'(0) = 0$

Exercise In the space below, sketch the solutions to the initial value problems from the previous example, i.e. $y'' + y = 4\delta(t - 2\pi), \quad y(0) = 1, y'(0) = 0$ and
$y'' + y = 4\delta(t - 2\pi), \quad y(0) = 0, y'(0) = 0$