Differential Equations

Math 341 Spring 2005 — MWF 8:30 - 9:25am Fowler North 2

http://faculty.oxy.edu/ron/math/341

Class 27: **Wednesday April 6**

TITLE *Derivatives of Laplace Transforms and Laplace Transforms of Integrals*
CURRENT READING Zill 7.4

Homework Set #11
Zill, Section 7.3: 3*, 7*, 15*, 22*, 39*, 43* *EXTRA CREDIT 49-54*
Zill, Section 7.4: 1*, 2*, 19*, 27*,33*, 41* *EXTRA CREDIT 45, 49*
Zill, Section 7.5: 3*, 9*
Zill, Chapter 7 Review: 25*,26*,27*,28*,29* *EXTRA CREDIT 37*

SUMMARY
We will look at the derivative of a Laplace Transform and introduce the concept of **convolution**.

1. Derivatives of Laplace Transforms

EXAMPLE Show that $\frac{d}{ds}F(s) = -\mathcal{L}[tf(t)]$ and $\frac{d^2}{ds^2}F(s) = \mathcal{L}[t^2 f(t)]$.

THEOREM When $F(s) = \mathcal{L}[f(t)]$, and $n = 0, 1, 2, \ldots$ $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$

Exercise We now have TWO different ways to show that $\mathcal{L}^{-1}[-te^{at}] = \frac{1}{(s-a)^2}$

EXAMPLE Solve $x'' + 16x = \cos(4t), \quad x(0) = 0, \quad x'(0) = 1$ using Laplace Transforms.

2. Products of Laplace Transforms

DEFINITION: **convolution**

If two functions $f(t)$ and $g(t)$ are piecewise continuous on $[0,\infty)$ then **the convolution of** f **and** g, usually denoted $f*g$ is defined to be $\int_0^t f(\tau)g(t-\tau)d\tau$. Note: this "product" is a function of t. The use of the "*" symbol is deliberate, since the convolution operation has these familiar properties:

THEOREM: properties of convolution

If f, g and h are piecewise continuous on $[0,\infty)$, then

I. $f*g=g*f$ (Commutative)

II. $f*(g+h)=f*g+f*h$ (Distributive Under Addition)

III. $f*(g*h)=(f*g)*h$ (Associative)

IV. $f*0=0$

THEOREM: **The convolution theorem** If f and g are piecewise continuous on $[0,\infty)$ and of exponential order so that $F(s)=\mathcal{L}[f(t)]$ and $G(s)=\mathcal{L}[g(t)]$ then$\mathcal{L}[f*g]=F(s)G(s)$

Corollary

$\mathcal{L}^{-1}[F(s)G(s)]=f*g$.

EXAMPLE The convolution theorem allows us to find inverse Laplace Transforms without resorting to partial fractions. For example, show that $\mathcal{L}^{-1}\left[\frac{k}{s^4+k^2s^2}\right]=\frac{kt-]sin(kt)}{k^2}$ by using the Convolution Theorem.

Exercise Evaluate $\mathcal{L}\left[\int_0^t e^{\tau}\sin(t-\tau)d\tau\right]$.

3. Laplace Transform of an Integral

We can use the Convolution Theorem with $g(t) = 1$ and show that $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \dfrac{F(s)}{s}$

NOTE

Multiplication of $f(t)$ by t involves **differentiation** of its Laplace transform $F(s)$ in s
Division by s of $F(s)$ involves **anti-differentiation** of the its Inverse Laplace Transform

Exercise Find $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+1)}\right]$ and $\mathcal{L}^{-1}\left[\dfrac{1}{s^2(s^2+1)}\right]$ and $\mathcal{L}^{-1}\left[\dfrac{1}{s^3(s^2+1)}\right]$

4. Volterra Integral Equations

A **Volterra integral equation** or **integro-differential equation** is an equation where the unknown function $f(t)$ (and/or $f'(t)$) appears on one side of the equation *and* in an integral on the other side, i.e. $f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau$

EXAMPLE **Zill, page 309, Example 5**. Solve $f(t) = 3t^2 - e^{-t} - \displaystyle\int_0^t f(\tau)e^{t-\tau}d\tau$

Exercise **Zill, page 313, HW #46**. Solve $y'(t) + 6y(t) + 9\displaystyle\int_0^t y(\tau)d\tau, \quad y(0) = 0$

THEOREM: **Laplace Transform of a Periodic Function**

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then $\mathcal{L}[f(t)] = \dfrac{1}{1-e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)dt$

EXAMPLE Let's derive this above formula.

Exercise Find $\mathcal{L}[f(t)] = F(s)$ where $f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & 1 \leq t < 2 \end{cases}$ and $f(t+2) = f(t)$.

Application Let's find the Laplace Transform of the Unit Triangle Wave of period 2. (See **Zill, Page 314, HW#49-54).**