Differential Equations

Class 26: Monday April 4

TITLE *Translations and The Laplace Transform*
CURRENT READING Zill 7.3

Homework Set #11
Zill, Section 7.3: 3*, 7*, 15*, 22*, 39*, 43* *EXTRA CREDIT 49-54*
Zill, Section 7.4: 1*, 2*, 19*, 27*,33*, 41* *EXTRA CREDIT 45, 49*
Zill, Section 7.5: 3*, 9*
Zill, Chapter 7 Review: 25*,26*,27*,28*,29* *EXTRA CREDIT 37*

SUMMARY
We will look at what happens when the independent and dependent variables in the Laplace transform are translated (or shifted).

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1. Translation in s

THEOREM: **First Translation Theorem**
If $F(s) = \mathcal{L}[f(t)]$ and a is any real number, then $\mathcal{L}[e^{at}f(t) = F(s-a)$. Sometimes the notation $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]|_{s\to s-a}$ is used.

Corollary
The inverse of the First Translation Theorem can be written as $\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t)$.

Exercise Given that $\dfrac{2s+5}{(s-3)^2} = \dfrac{2}{s-3} + \dfrac{11}{(s-3)^2}$, compute $\mathcal{L}^{-1}\left[\dfrac{2s+5}{(s-3)^2}\right]$. (HINT: recall that $\mathcal{L}^{-1}\left[\dfrac{1}{s^2}\right] = t$)

EXAMPLE Compute $\mathcal{L}^{-1}\left[\dfrac{s/2+5/3}{s^2+4s+6}\right]$

EXAMPLE **Zill, Example 3, page 295.** Let's use Laplace Transforms to show that the solution of $y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$ is $y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t}$.

2. Translation in t

DEFINITION: **Heaviside function**

The **unit step function** or **Heaviside function** $\mathcal{H}(t)$ is defined to be **0** when its argument is less than zero and **1** when its argument is greater than or equal to zero. Generally, it is written as $\mathcal{H}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

GROUPWORK Confirm that $f_1(t) \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$
can be written as $f(t) = g(t) - g(t)\mathcal{H}(t-a) + h(t)\mathcal{H}(t-a)$

How would you combine Heaviside functions to represent the following function?

$$f_2(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$

THEOREM: **Second Translation Theorem**

If $F(s) = \mathcal{L}[f(t)]$ and $a > 0$ is any positive real number, then $\mathcal{L}[f(t-a)\mathcal{H}(t-a)] = e^{-as}F(s)$.

It directly follows then that $\mathcal{L}[\mathcal{H}(t-a)] = \dfrac{e^{-as}}{s}$.

Corollary

$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\mathcal{H}(t-a)$

THEOREM: **Alternate form of the Second Translation Theorem**

It can be annoying to try and get the function which is multiplying the Heaviside function into the form $f(t-a)$ for use in the previous version of the Second Translation Theorem so a more useful results is: $\mathcal{L}[g(t)\mathcal{H}(t-a)] = e^{-as}\mathcal{L}[g(t+a)]$

EXAMPLE Compute and graph $\mathcal{L}^{-1}\left[\frac{s}{s^2+9}e^{-\pi s/2}\right]$ on $t \geq 0$.

Exercise **Zill, page 299, Example 7**. Compute $\mathcal{L}[\cos(t)\mathcal{H}(t-\pi)]$

3. Application Problems/Examples/Exercises

Let's use our knowledge of Laplace Transforms to solve some otherwise difficult initial value problems and boundary value problems.

Application

Zill, page 303, #31. $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1$ where $f(t) \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & 2\pi \le t \end{cases}$

Application **Zill, page 301, #31.** $y'' + 2y' + y = 0, \quad y'(0) = 2, \quad y(1) = 2.$